Point-pattern analysis

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Outline

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What is a point pattern?

- A set of (georeferenced) point **locations** within a defined **region**
- Resulting from a **data-generating process** (DGP) operating over the region
- A DGP that results in the location of points is called a **point process**

  *"a stochastic mechanism which generates a **countable** set of events" [7]*

- Details
  - Lines and polygons may be reduced to "points", e.g., by centroid, and treated as points
  - The **region must be carefully defined**, otherwise most statistics are distorted

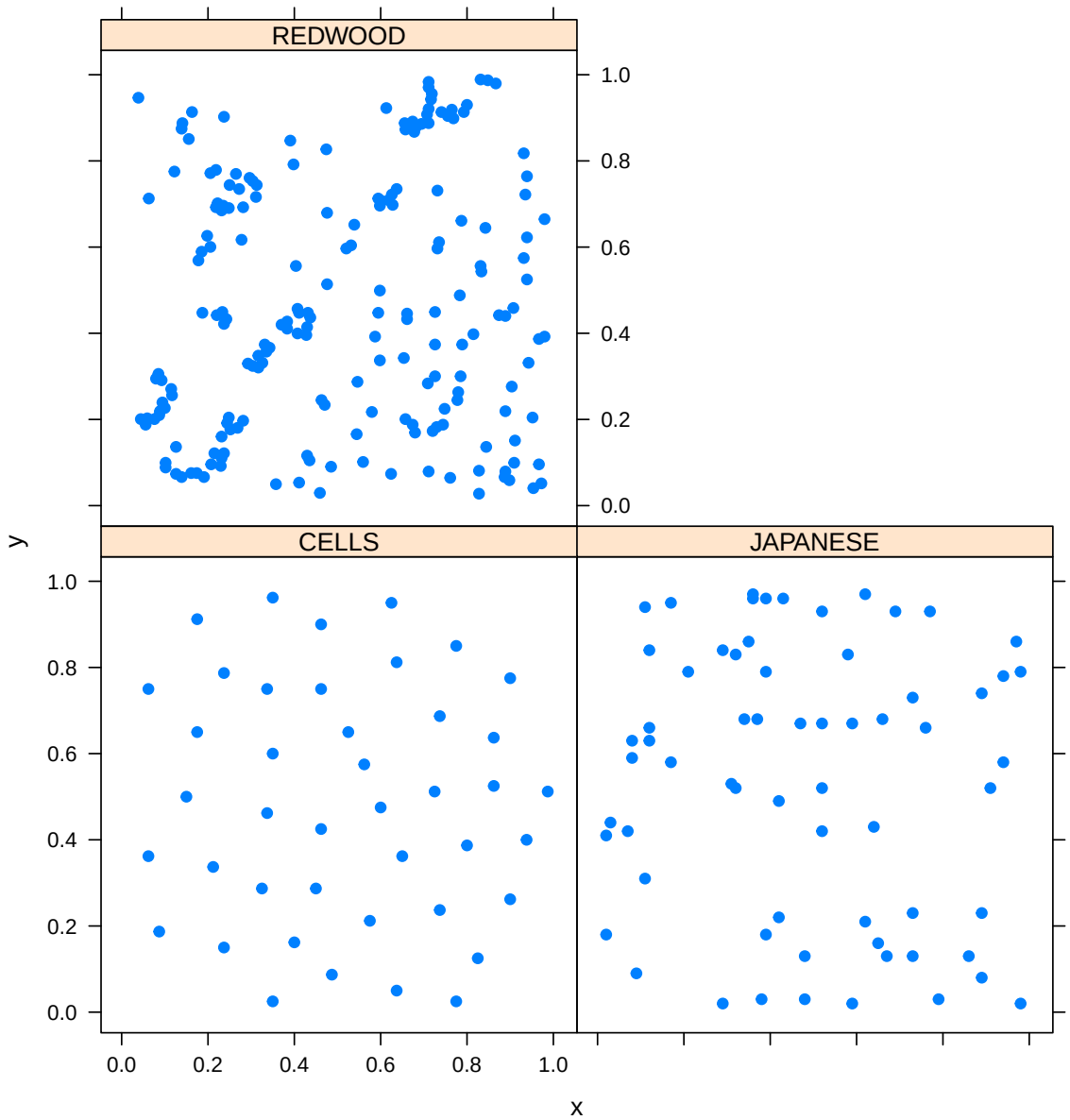
Examples

- Position of (one species of?) trees on a landscape
- Traces of meteor strikes
- Geomorphic features (reduced to points)
  - e.g., drumlin fields; Carolina bays
- Location of crime incidents
  - single type of crime, interaction between types of crime, interaction with point features, e.g., banks
- Distribution of grazing animals in a field
  - could study the evolution of the point-pattern over time
  - interaction between two species
- Bomb strikes around a target

Examples used in ASDAR text

- Redwood trees in a forest plot
- Cell centres on a microscope slide
- Japanese pines in a forest plot

Point-patterns on a unit square



Example: Redwood trees

source: Strauss [17]

Hypotheses to test:

> *" It was felt that the seedlings would be **scattered fairly randomly**, except that **a number of tight clusters** would form around some of the ...stumps present in the plot. A **discontinuity in the soil**, very roughly demarked by the diagonal line in the figure, was expected to cause a **difference in clustering** behaviour between regions I [upper left] and II [lower right]. Moreover, almost all the ...stumps were situated in region II"*

area about 50 x 50 m = 0.25 ha

> *[1.8 m] "which was thought to be very roughly the range at which a pair of seedlings could 'interact' "*

So maximum density $\approx$ 1 500 trees

Example: Cell centres

- Hypothesis to test: cells form a regular pattern

Example: Japanese pines

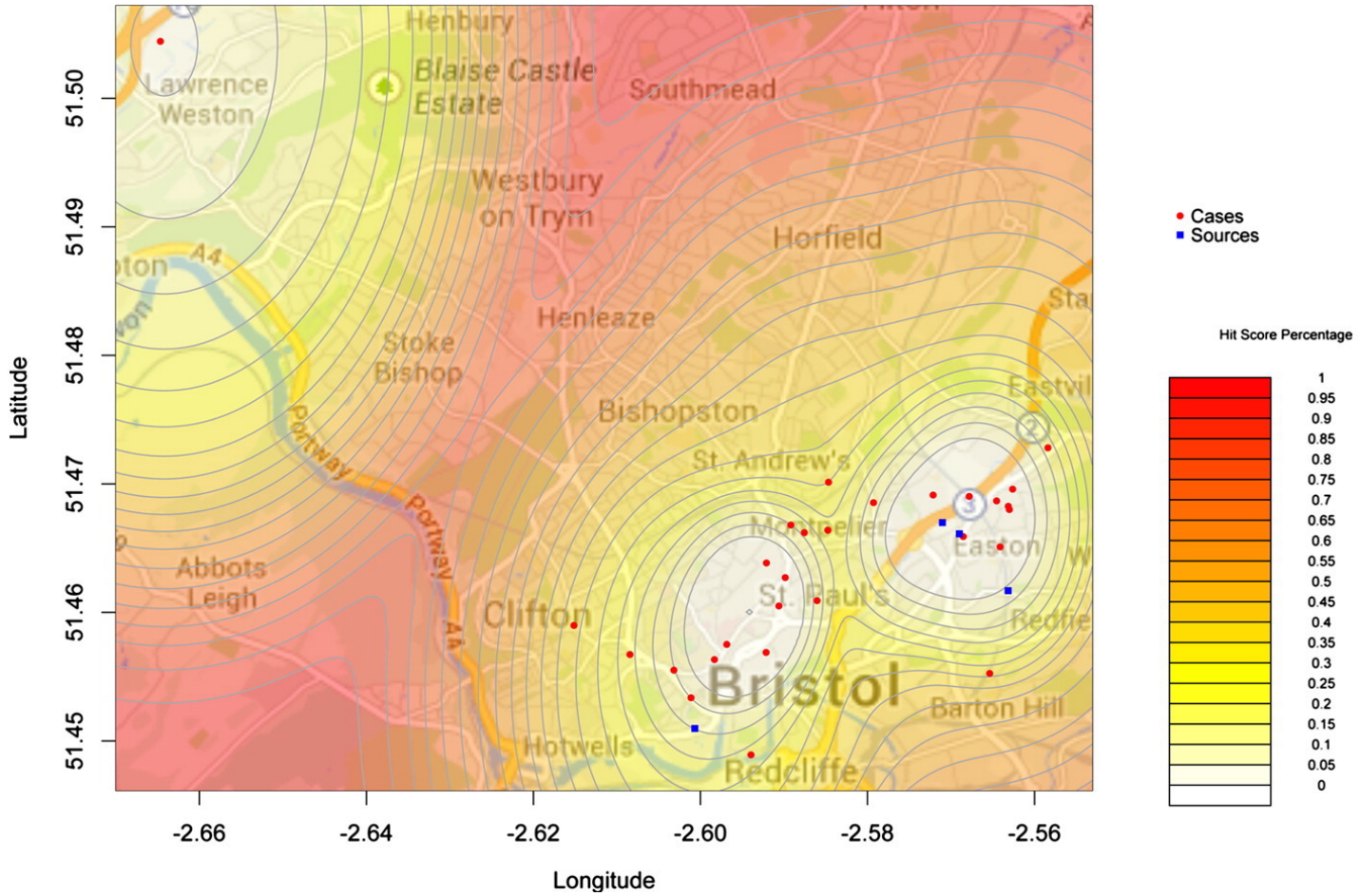
- Hypothesis to test: trees are distributed randomly: no attraction, no repulsion, no regular order

Example: Drumlin field

Cayuga County, NY tops of drumlins can be considered as "points"
Random? Clustered? Regular pattern? DGP is Laurentide Ice Sheet

See also `https://rsbivand.github.io/ECS530_h21/ECS530_211119.html#Point_patterns`

Example: "crime" vs. suspect



source: Hauge, M. V. *et al.* (2016). *Tagging Banksy: using geographic profiling to investigate a modern art mystery*. **Journal of Spatial Science**, 1–6. `http://doi.org/10.1080/14498596.2016.1138246`

Why analyze point patterns?

- We are interested in the **process** which produced the **pattern**
- We can only **observe** the pattern
- We want to **infer** the **spatial data-generating process** (sDGP) – **why** are the points located as they are?
  - How? make a **model** based on hypotheses that "best" fits the observed pattern
  - Allows to confirm / reject / modify a geomorphological, ecological or social **theory**
- We may want to assign a **density** to every location in a region
  - probability of occurence, normalized by area
- We may want to **aggregate** counts / densities over some area

Outline

Stochastic processes

- An observed point pattern is a **realization** of some **stochastic process**.
- The points can be labelled as $\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$, and the number of points in any region A is $N(A)$.
- What we observe is **one realization** of a process,
- If we could "rewind" time and run the same process, we should see the **same statistical properties** resulting from the process, but a **different empirical point pattern**.
- As analysts we want to **infer** the process from the one realization.

Process orders

1. **First**-order
   - **one** set of points, each point as **separate occurrences** from some **stochastic process**
   - point distribution (observed spatial **density**) *suggests* spatial (in-)homogeneity of process **intensity**
     - ⋆ completely spatially-**random** (CSR)?
     - ⋆ **clustered**?
     - ⋆ **dispersed**?
     - ⋆ **regularly-spaced**?
2. **Second**-order process
   - **interaction** between (positions of) points
   - One or more sGDP, if multiple with with interaction between them
   - model **random** distribution (no interaction), vs. **attraction**, vs. **repulsion**
3. both can (partially) depend on spatial **covariables**
   - e.g., regional trend or environmental factors
   - Strauss processes, see below

First-order stochastic processes

- Denote small regions in $A \subset \mathbb{R}^2$
- Define the **first-order point intensity** of the process as:

$$\lambda(\mathbf{x}) = \lim_{|d\mathbf{x}| \to 0} \frac{E[N(d\mathbf{x})]}{|d\mathbf{x}|}$$

- This can be computed at any radius $|d\mathbf{x}|$, but the theory requires a **point** density.

Stationarity and Isotropy

- Divide a region A into a set of smaller regions $\{A_i : i = 1 \ldots k\}$ each with observed points $N(A_i)$.
- Examine what happens if we **translate** or **rotate** the points, thereby resulting in different $N(A_i)$.
- If the joint distribution does not change, the process is *stationary* and *isotropic*.
  - i.e., the process operates identically over the region, with no directional effects nor sub-regions with different intensity.
- *If* this is the case, $\lambda(\mathbf{x}) = \lambda = \frac{E[N(A)]}{|A|}$, i.e., the same intensity everywhere.
- The second-order density reduces to $\lambda_2(h) = \lambda_2(||\mathbf{x} - \mathbf{x}||)$
  - i.e., a function of their **separation distance** as vectors of coordinates.
- The covariance density is then $\lambda_2(h) - \lambda^2$.

Poisson point process

- Example of a Data Generating Process (DGP)
- Named for the Poisson statistical distribution

$$f(k;\lambda) = \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

- The number of points in a region is a **random variable** with a Poisson distribution, intensity λ

  homogeneous λ the same everywhere
  inhomogeneous λ can vary over space (cluster process)
- **Locations** of points in the region, given the intensity, are **completely random**.

"Hard core" Poisson processes

- A Poisson process but there is a **hard core** distance, within which no two points can be closer to each other
- Often used when the "point" has a certain size
  - Example: mature trees in a savannah, canopies do not overlap, so locations of tree trunks can not be closer than the canopy width

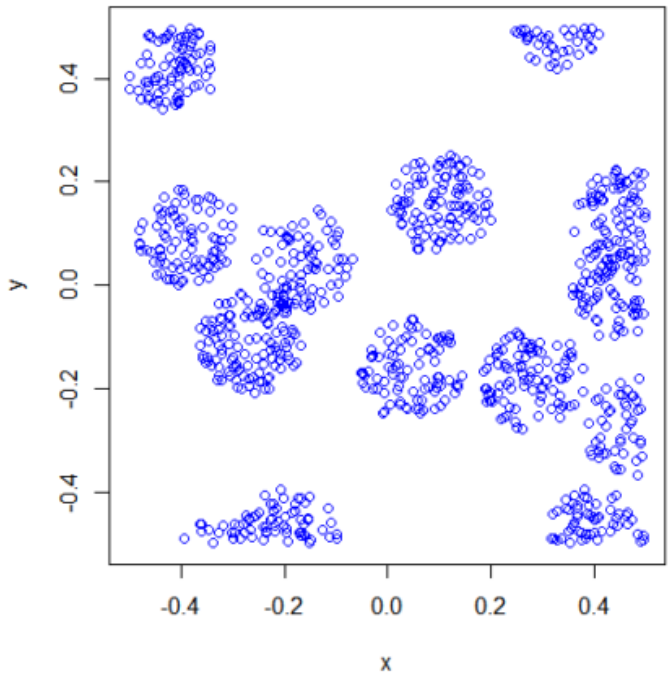
credit: Hanspeter.Baumeler; CC BY-SA 4.0
Klein Nossob (Namibia); 24° 33' 18.73" S, 19° 47' 12.24" E

Cox point process

- A type of Poisson point process, but ...
- ...the **intensity**, i.e., Poisson parameter $\lambda(s)$, **varies** across the region ...
- ...according to some **external** stochastic process
  - e.g., geographic trend in climate; patches of different soils with different suitability for a species
- The cause may not be known, but the result can be analyzed

Matérn cluster process

- One of many processes studied by Matérn, a Swedish forestry statistician [12]
- (1) Randomly-located (Poisson process) points tend to (2) form random (Poisson process) **clusters** around these seeds
- Poisson intensity of clustered points is higher than the seed points
- Example: locations of rare species are random within a landscape, but each "location" is a cluster (local seed dispersal? favourable soil conditions?)



Strauss process

- There is either **clustering** or **repulsion** between points, but this process is **homogeneous** across the region
- $\lambda(u, \mathbf{x}) = \beta\gamma^{t(u,\mathbf{x})}$
  - $\lambda(u, \mathbf{x})$: intensity of pattern $\mathbf{x}$ at location u;
  - β: overall homogeneous intensity;
  - γ: **interaction** parameter $0 \leq \gamma \leq 1$
  - $t(u, \mathbf{x})$: the number of points closer than the **interaction radius** r
- Interpretation:
  - $\gamma = 0 \Longrightarrow \lambda = 0$: no chance of finding another point;
  - $\gamma < 1$: chance of a second point is reduced;
  - $\gamma = 1$ equivalent to a Poisson process (CSR)
- Specify r, fit γ from observed pattern

Second-order stochastic processes

- Consider interactions between occurrences **x** and **y**.
- i.e., the location of one point may affect the location of another, either to **aggregate** ("attract") or **disperse** ("repulse").
  - Must then infer the mechanism.
- Express this as the **second-order** intensity function:

$$\lambda_2(\mathbf{x}, \mathbf{y}) = \lim_{|d\mathbf{x}| \to 0, |d\mathbf{y}| \to 0} \frac{E[N(d\mathbf{x})][N(d\mathbf{y})]}{|d\mathbf{x}||d\mathbf{y}|}$$

- In other words, the density of the two occurrences at the same location.
- These have a **covariance density**, i.e., to what degree the two sets co-locate:

$$\gamma(\mathbf{x}, \mathbf{y}) = \lambda_2(\mathbf{x}, \mathbf{y}) - \lambda(\mathbf{x})\lambda(\mathbf{y})$$

Second-order "Hard core" Poisson processes

- A Poisson process but there is a **hard core** distance, within which no two points, **one from each process** can be closer to each other
- Represents **repulsion** from one set of points to another
  - Example: one tree species inhibits the growth of another

Outline

Intensity

- The simplest measure: how many points on average per unit area
  - observed point **density** $\leftarrow$ **intensity** of the process that produced them
- **homogeneous** process: $\lambda(s) = \lambda = n/|A|$
  - s = spatial location; n = number of points; $|A|$ = area
  - does not vary over the area; **expected** value the same everywhere
- **inhomogenous** process: $\lambda(s)$ varies over the area
- depends on the **scale** at which we examine it (the **bandwidth**)
  - **broad** (wide): all points are taken together and the process is by definition homogeneous
  - **fine** (narrow): random fluctuations lead to different intensity estimates, even of same process
- **narrow** bandwidth $\rightarrow$ "spiky" map; **wide** bandwidth $\rightarrow$ (over-?)smooth map
- try to **match** bandwidth with the scale of the sDGP

Kernel density estimate : objective

- We suppose the process is **in**homogeneous
- We want to estimate the **intensity** at all locations
  - i.e., what is the area-normalized **probability** of a point at each location?
- Only uses the point-pattern itself (no covariables, no trend)
- Non-parametric (no model of the underlying process)
- This can lead to **hypotheses** about the spatial **process** (sDGP)
  - overlay on presumed covariable to see if they match
  - e.g., vegetation density vs. soil type polygons

Kernel density estimation: concept

Simplest is the **spherical** kernel; in 2D this is **circular**:

$$\hat{\lambda}(x) = \frac{N(b(x,h))}{\pi h^2}$$

where:

- $b(x,h)$ is a disc of radius h centred at x
  - $\hat{\lambda}(x)$ is estimated density per unit square
  - h is the **bandwidth**
  - larger $\rightarrow$ smoother estimates as kernel moves across the map
- N is the number of points in the disc
- denominator πh^2 is the area of that circle, normalizes the density

Kernel density estimation : with kernel function

This simple spherical count can be generalized with a **kernel function** that gives more weight to "nearby" portions of the disc:

$$\hat{\lambda}(x) = \frac{1}{h^2} \sum_{i=1}^{n} \kappa \left(\frac{||x - x_i||}{h} \right) / q(||x||)$$

where:

- $|| \cdot ||$ is the signed **norm**, usually the Euclidean distance between the target position (centre of kernel) and an observed point
- $\kappa(u)$ is a bivariate, symmetric **kernel function** of $u = ||x - x_i||/h$ (see next slide)
- $q(||x||)$ is an **edge-correction** factor (unobservable points outside the boundary)
- h is the **bandwidth**

Smoothing kernel

Example: the **quartic** (a.k.a. **biweight**) kernel:

$$\kappa(u) = \begin{cases} \frac{3}{\pi}\left(1 - ||u^2||\right)^2 & \text{if} \quad u \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

- as points are further away from the centre of the kernel, they get less weight in the density estimate
- u is signed according to the coördinate system, but then squared
- outside the normalized bandwidth $|u| \geq 1$ any points are *not* included in the density for a given location
- this is controlled by the h parameter

Choice of bandwidth

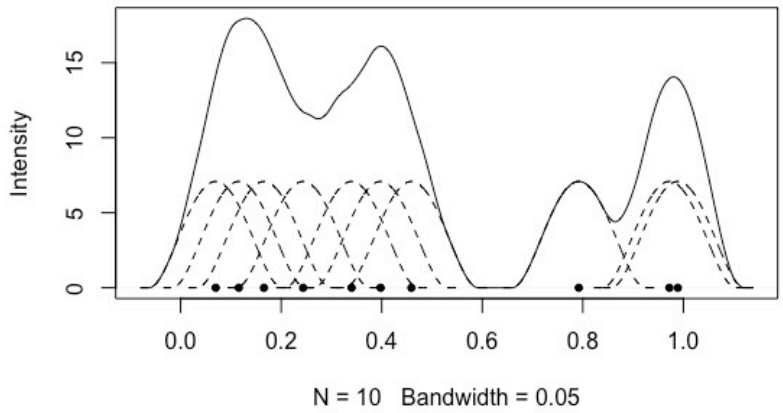
Which bandwidth "best" represents the (in)homogeneity of the point process?

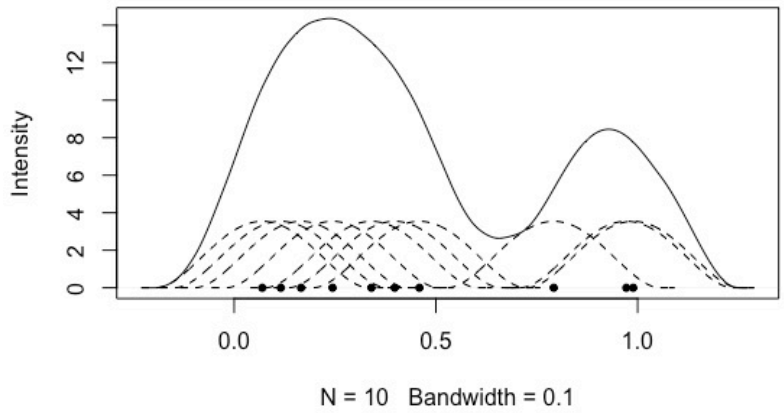
> *"For any kernel function a **small** value of h may result in an estimated surface $\hat{\lambda}(x)$ that is **too spiky**, whereas a **large** h leads to smoother surfaces but may **ignore local features** of $\hat{\lambda}(x)$.*
>
> *"**No simple recipe** for the choice of bandwidth exists ….*
>
> *"**Background information** on the objects that form the pattern, such as dispersal distances for plants, might inform bandwidth choice …*
>
> *"[In] the absence of this the user should simply consider a number of values of h and choose the one that gives the **most plausible result** in the **specific context**." – Illian et al. [10, p. 115]*

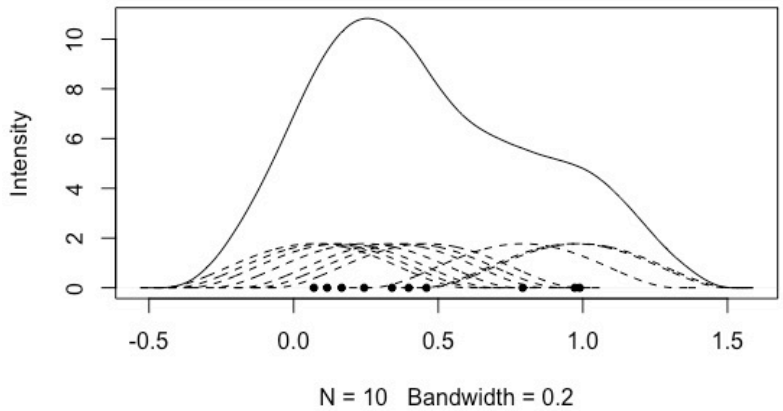
Choice of bandwidth – relation to application

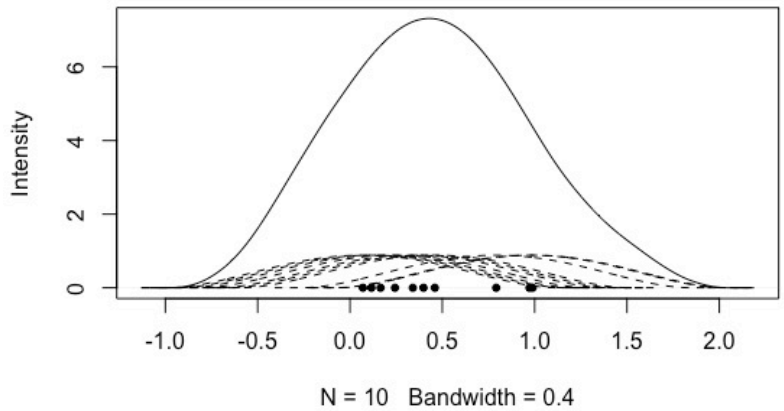
- How is "density" **perceived** in the application context?
  - Example: "dense" vegetation as cover for small animal (e.g., fox) vs. large animal (e.g., deer) – what is the radius that has to be "dense" before the animal feels secure using it as cover?
- What is a **realistic maximum density**?
  - If the process were homogeneous across the whole study area, what would be the maximum number of points per unit area? The kernel density at any point should not exceed this.
  - Example: mature trees with non-overlapping main canopies, radius ≈ 3 *m*, so no more than about $10000/(\pi \cdot 3)^2 \approx 350$ trees per hectare.
  - Example: redwood seedlings ≈ 6000 per hectare, ≈ 1500 in the example plot (Strauss)

Effect of bandwidth: 1D example





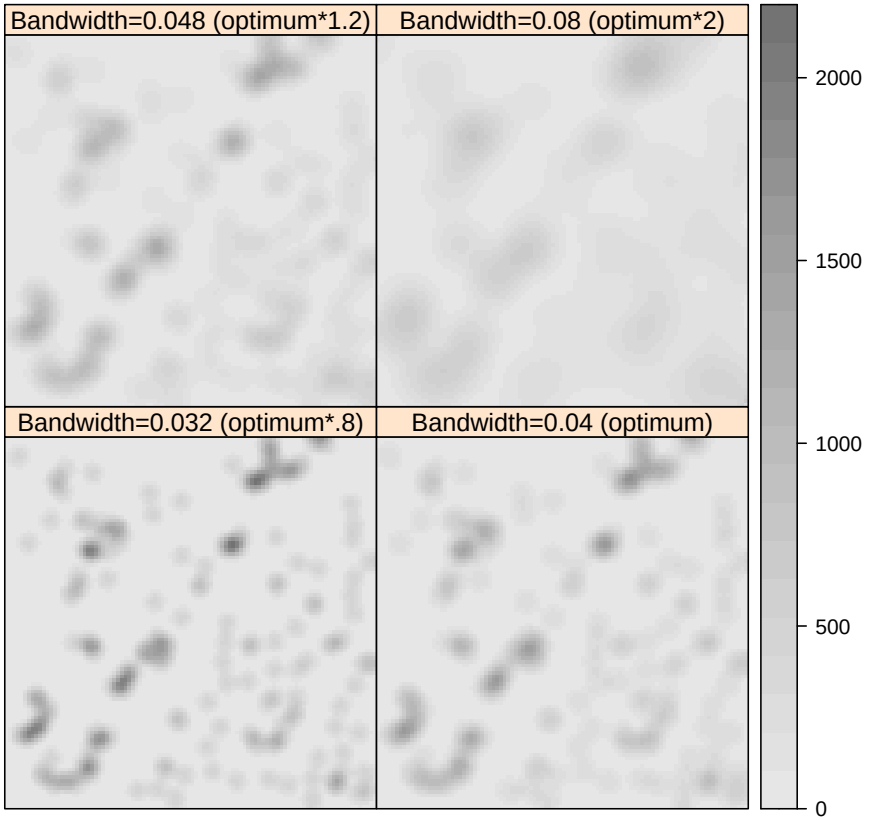


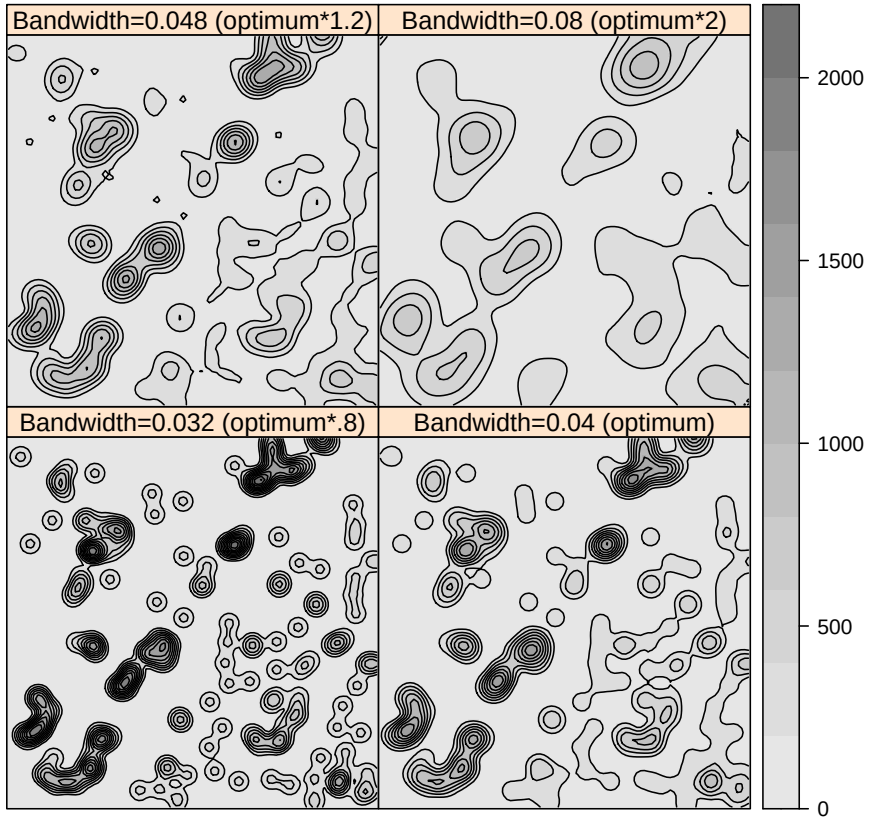


$\lambda = 10$; solid line: biweight kernel density

dashed lines: contribution of each point to density

Effect of bandwidth: 2D example: redwood trees





– different bandwidth → different estimate;
– overall density 195 trees in the unit square
– density 2200 (unit)$^{-1}$ in "hottest" spot / narrowest bandwidth → unrealistic (> 1500)

Choice of bandwidth – Diggle's method

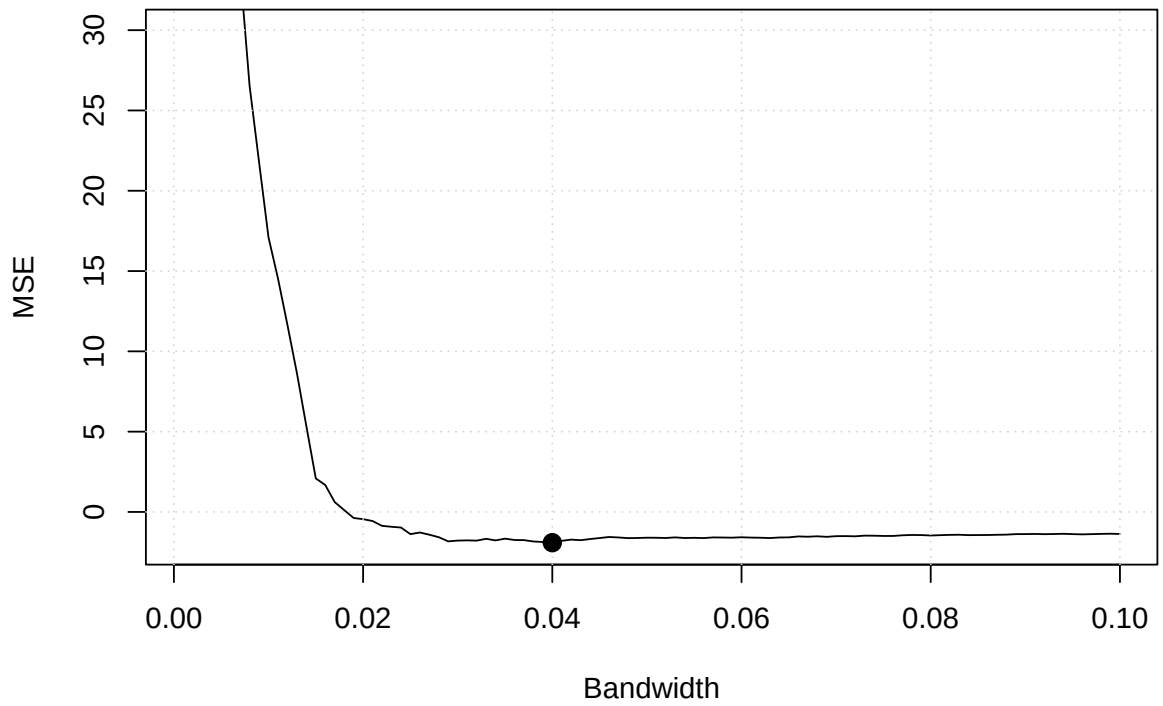
Diggle [8] provides a bandwidth estimation method for the Cox process: Minimize the mean square error (MSE) of the kernel smoothing estimator vs. actual counts.

$$\mathrm{MSE}(r) = E\left\{\left[\hat{\lambda}_r(x) - \lambda_r(x)\right]^2\right\}$$

- Try a series of bandwidths
- For each, compute the kernel density at each grid point, compare to counts
- Summarize across area
- Graph MSE vs. bandwidth, look for first minimum

Simpler: $h = 1/\sqrt{n}$.

Diggle's method – redwood trees



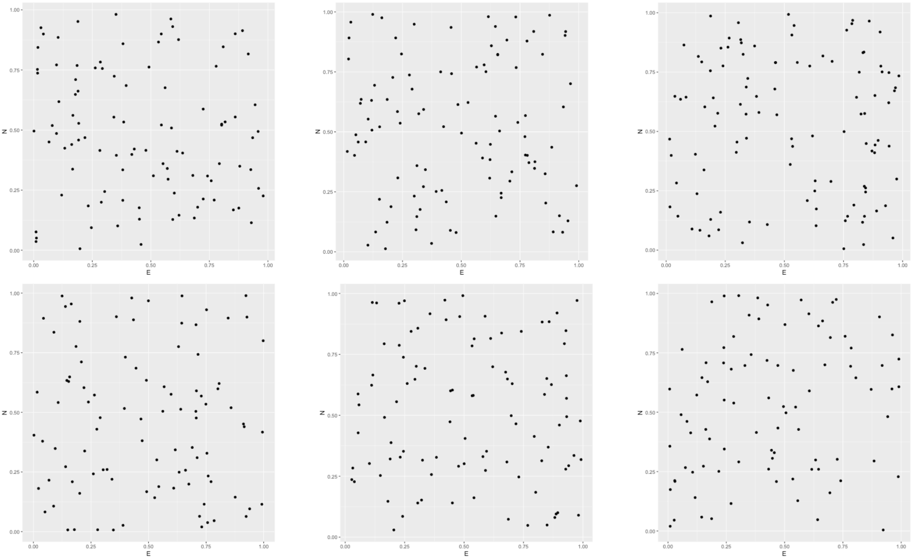
0.04 on unit square $\approx 0.04 * 50 = 2$ m at field scale; Strauss used 1.8 m

Homogeneous density: $1/\sqrt{195} = 0.072$.

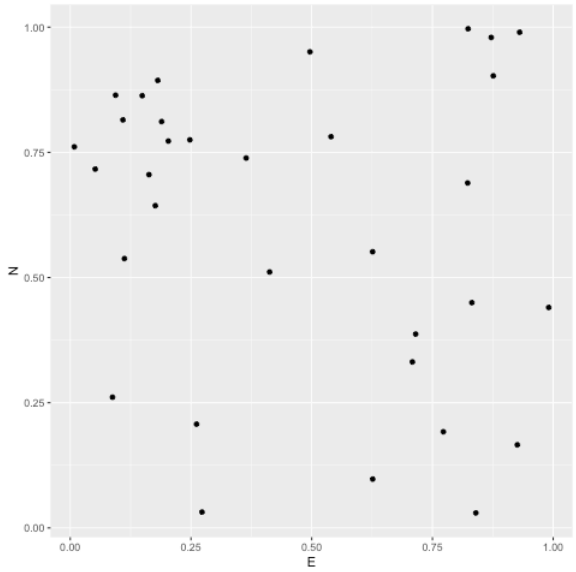
Complete spatial randomness (CSR)

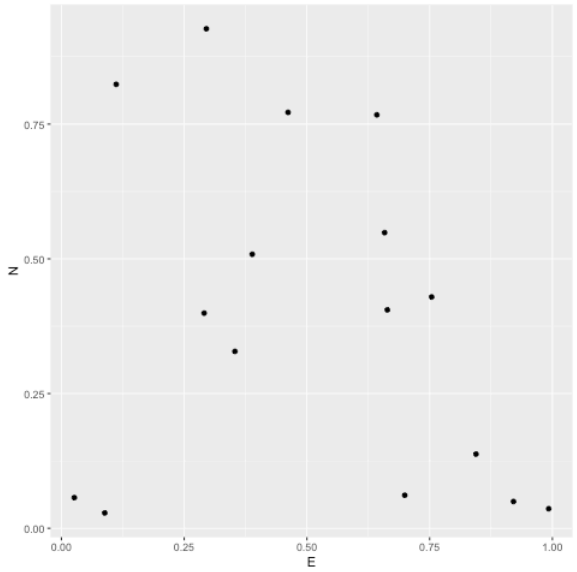
- The simplest **null hypothesis** of spatial distribution
- Points are distributed randomly; **no interaction**
  - no **clustering** (attraction, preferential conditions ...)
  - no **dispersion** (repulsion, limited resource in area ...)
- Produced by a (spatially) **homogeneous Poisson process**
- Constant **intensity** $\lambda(s) = \lambda > 0, \forall s \in A$
- The **probability** of any number of points in the same-sized region is the same, across the entire field.
- **Simulate** with the `spatstat.random::rpoispp` function
  - argument is λ (process intensity)

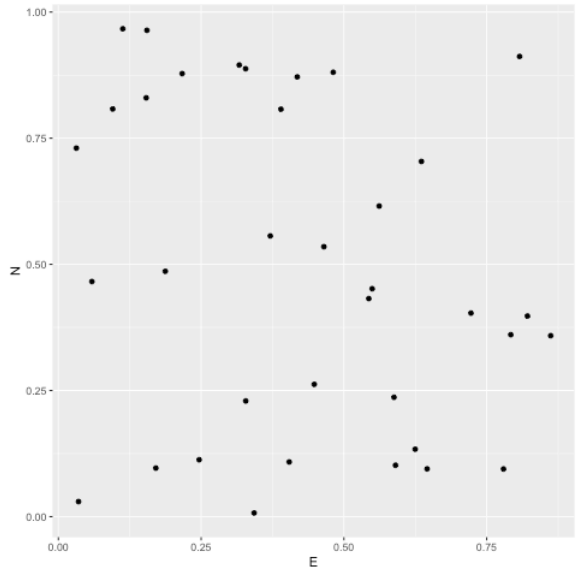
CSR: 100 random points in a unit square

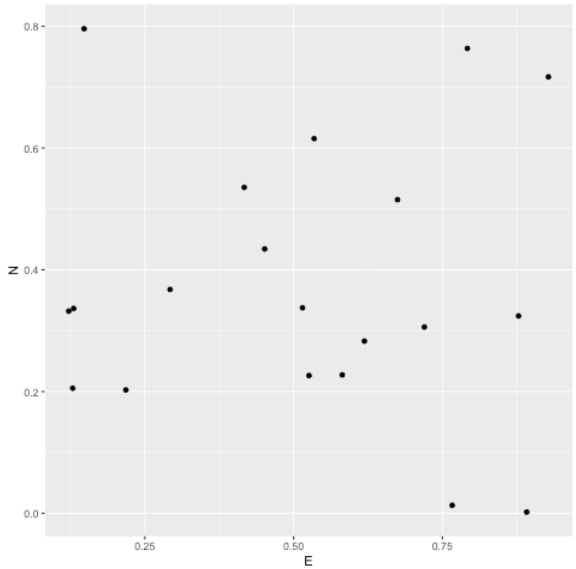


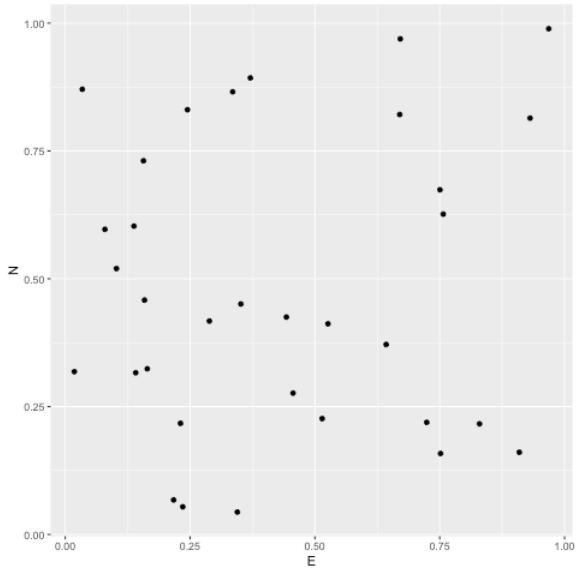
CSR: 30 random points in a unit square

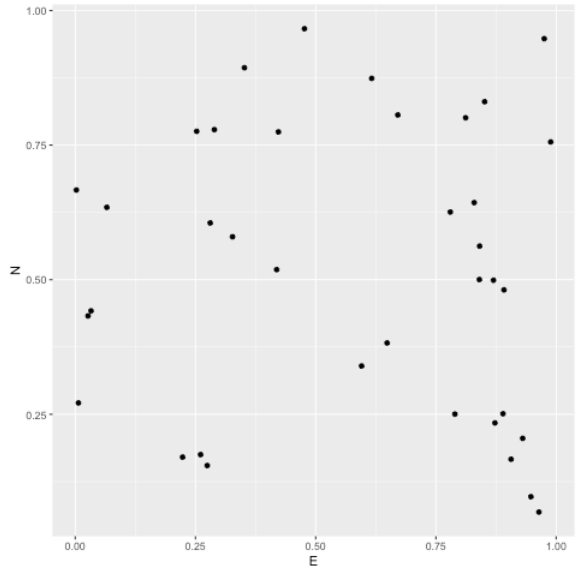












A naïve approach – spatially-discretized Poisson distribution ("quadrat analysis')'

- Spatially homogeneous **Poisson process**: count of "rare" events (points) in a discrete area follows the Poisson distribution
  - $f(k;\lambda) = \Pr(X = k) = (\lambda^k e^{-\lambda})/k!$
- a single parameter: process **intensity** $\lambda = \mu(X) = \sigma^2(X)$
  - i.e., the mean count, and its variance, are the intensity
- **Test** for Poisson process: **discretize** the area, compute λ, count the points, compare to Poisson distribution
  - Also known as **quadrat** analysis of a point pattern.
- Q: how fine a discretization? A: about half of the cells should have Poisson expectation 0

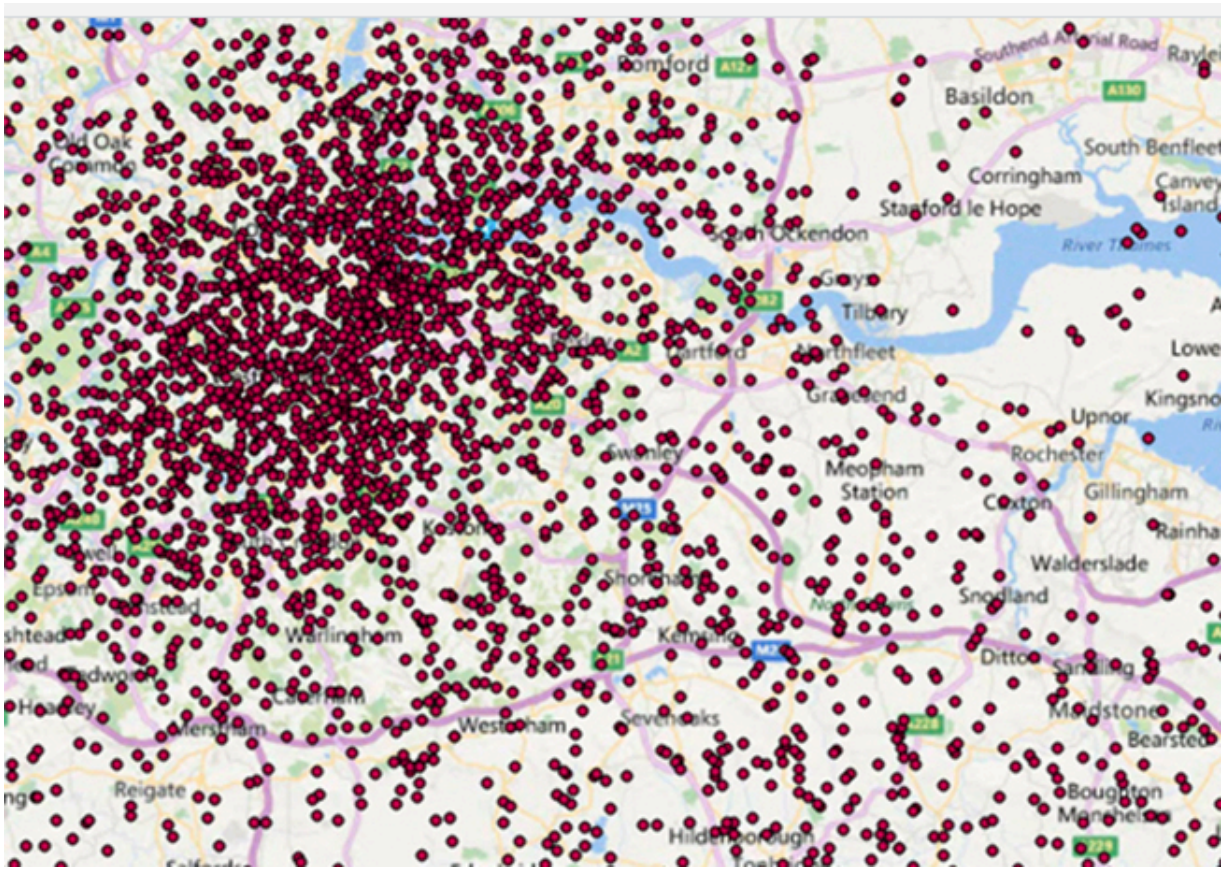
Example of Poisson counts: V-1 rocket strikes

- **point process**: WW2: German V-1[1] rockets aimed at England
  - the first cruise missile
  - later in WW2 replaced by the V-2 ballistic missle (direct precursor of US Redstone rocket used in early manned space programm)
    - ★ "once the rockets are up, who cares where they come down? / That's not my department, says Wernher von Braun."
      – Tom Lehrer, *Wernher von Braun* (1964)
- timed to reach their target by running out of fuel
- **point pattern**: clustered? dispersed? random?
  - inference about V-1 targeting system and objectives
  - **practical implications** for siting anti-aircraft guns, protecting industry, population, civil defence, fire brigades ...

[1] "*Vergeltungswaffe*" = "Vengeance Weapon"

Fieseler Fi 103 V-1 on a Walter catapult ramp at Sperleke/Éperlecque, Pas de Calais (F)
Credit: By Murgatroyd49 - Own work, CC BY-SA 4.0,
`https://commons.wikimedia.org/w/index.php?curid=97238674`

V-1 strikes point pattern – London area



Source: Metcalfe [13]

V-1 example: computations by Clarke (1946)

- source: Clarke [6]; see also Shaw and Shaw [16]
- selected an area in S London where "the theoretical mean density was not subject to material variation anywhere within the area examined", i.e., no reason to expect different attack intensity, e.g., due to target concentrations
- Discretize 144 km^2 area into $n = 576$ squares of 0.25 km^2
  - this corresponds to the bandwidth
- Poisson distribution:
  - Total hits 537, so intensity $\lambda = 537/576 = 0.932 \approx 1$
  - $f(0, 0.932) = e^{-0.932} = 0.39377$, i.e., about 40% of cells with no expected hits; $f(0, 0.932) \cdot 576 \approx 227$ grid cells
    - ⋆ this is probably why Clarke chose 0.25 km^2 cells
  - $f(2, 0.932) = (0.932^2 e^{-0.932})/2 = 0.171$; $f(2, 0.932) \cdot 576 = 98.54 \approx 99$ grid cells with exactly 2 expected hits

V-1 example: results

n	expected	actual
0	226.74	229
1	211.39	211
2	98.54	93
3	30.62	35
4	7.14	7
≥ 5	1.57	1

$\chi^2 = \sum \frac{(O-E)^2}{E} = 1.17$; $\Pr(\chi^2(4) > 1.17) = 0.88$

Conclusion: hits are *not* provably different from the null hypothesis of a homogeneous Poisson process; within the target area the distribution is CSR

The G function

- A more sophisticated approach for evaluating CSR and deviations from it
- Measures the distribution of distances from an arbitrary point to its **nearest neighbour**. The **empirical** function is:

$$\begin{aligned} d_i &= \min_j \{d_{ij}, \forall j \neq i \in S\}, i = 1, \ldots, n \\ \widehat{G}(r) &= \frac{\{\#d_i : d_i \leq r, \forall i\}}{n} \end{aligned}$$

  - this is the number (#) of points which have *at least one* neighbour within some threshold distance r, *normalized* by the total number of points n in the pattern S.
- This is a **continuous** function of r – no need to discretize

G function under CSR

The result of the homogeneous Poisson process is the **theoretical** function:

$$G(r) = 1 - e^{\lambda \pi r^2}$$

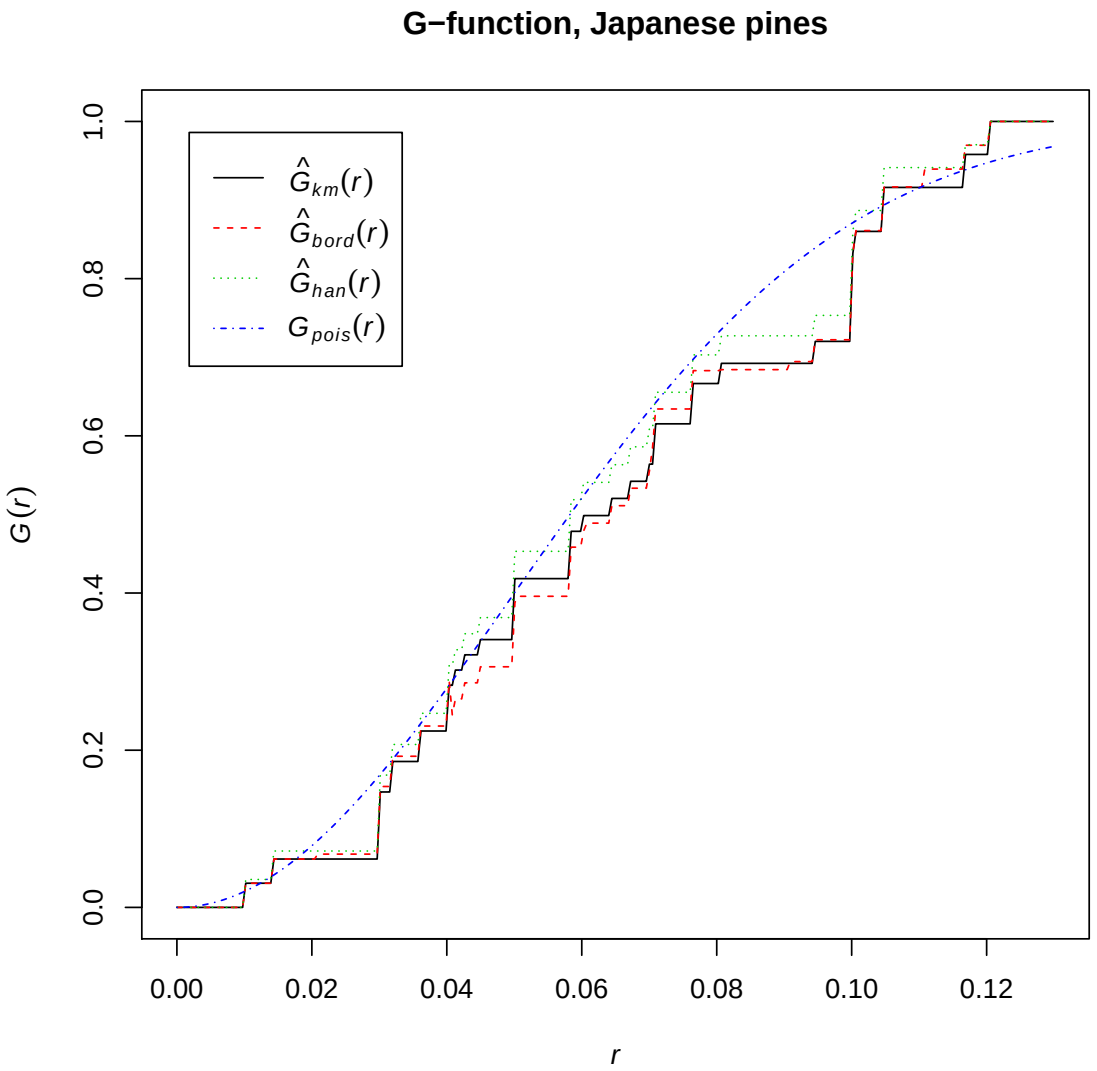
where λ is the process **intensity**, i.e., mean number of points per unit area.

Clustered patterns: $\widehat{G}(r) > G(r)$ (more nearby points than expected under CSR)

Dispersed (regular) patterns: $\widehat{G}(r) < G(r)$ (fewer ...)

These are all evaluated at *any* threshold (radius) r, can infer radius of clustering/dispersion

Example G function

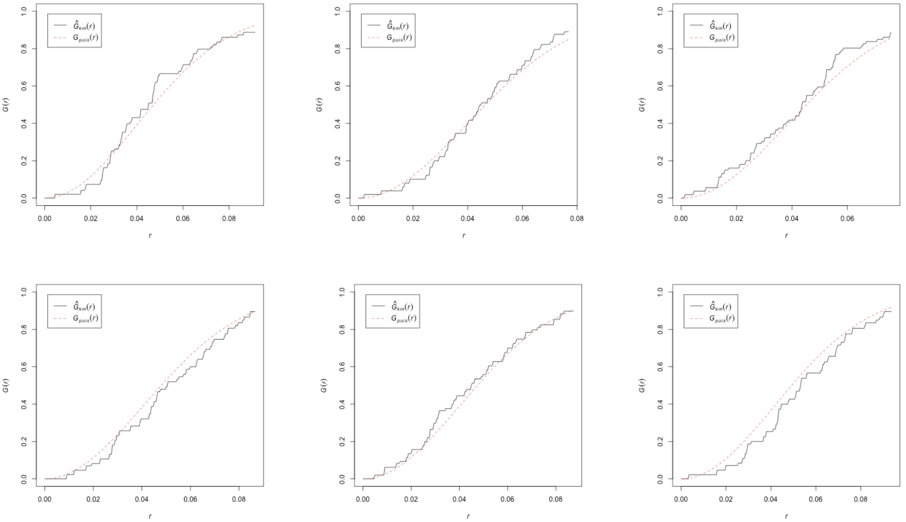


- G_{pois}: theoretical CSR
- G_{km}: empirical distribution
- G_{bord}, G_{han}: border-corrected empirical distributions

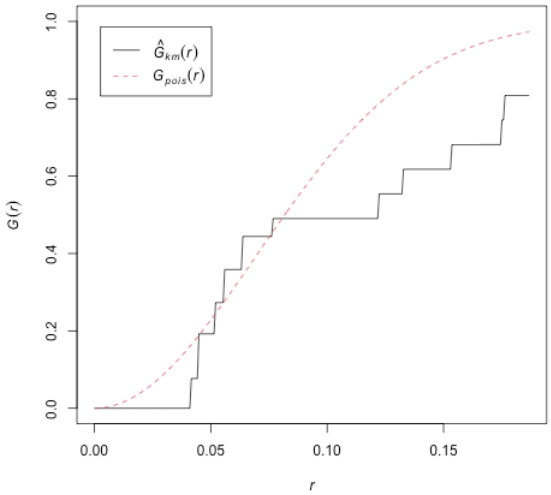
Example G function: interpretation

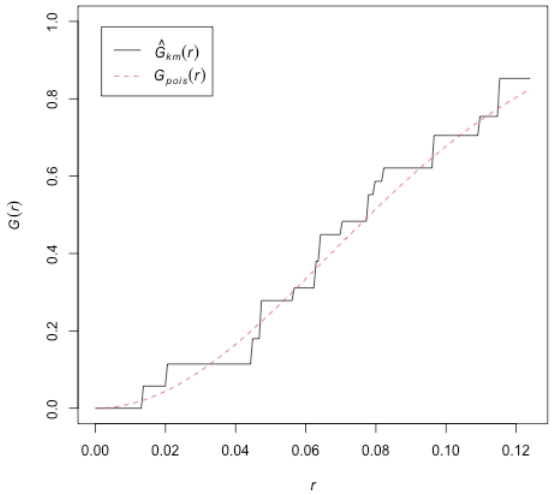
- All points have a nearest neighbour within 0.13 normalized units
  - compute inter-point distances with `nndist`
- Empirical closely follows theoretical CSR
- Some deviations (dispersal, below the theoretical line) near 0.02 and 0.10 normalized units
- The border has little effect

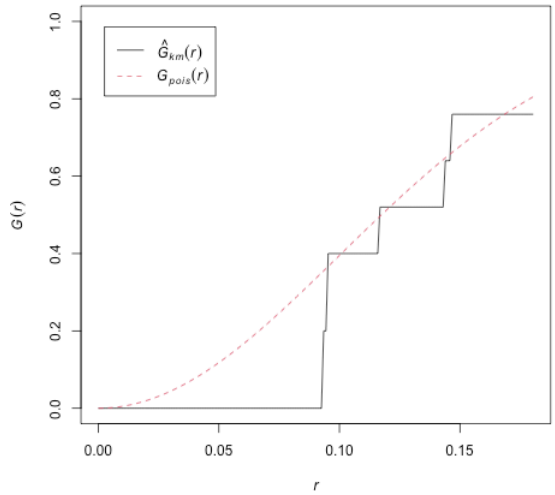
G-function: 100 random points in a unit square

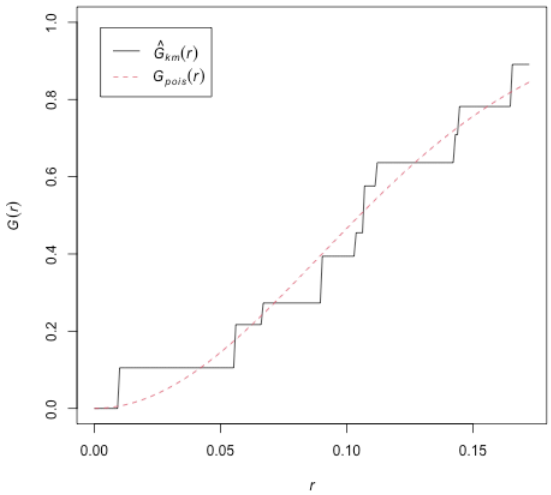


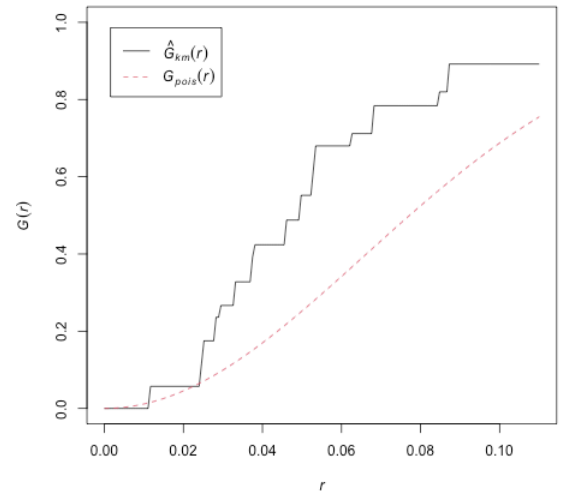
G-function: 30 random points in a unit square

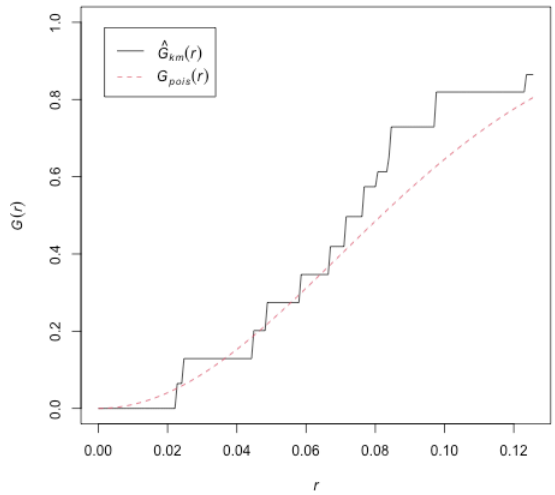










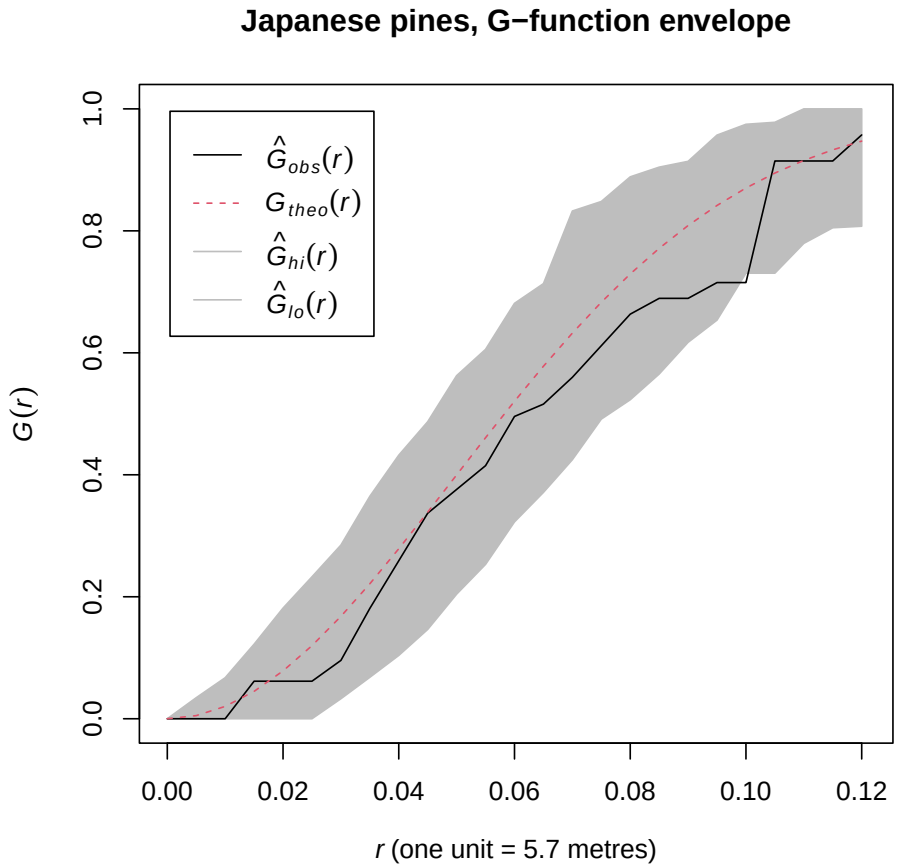


Envelope

A method to determine **confidence intervals** for the G function:

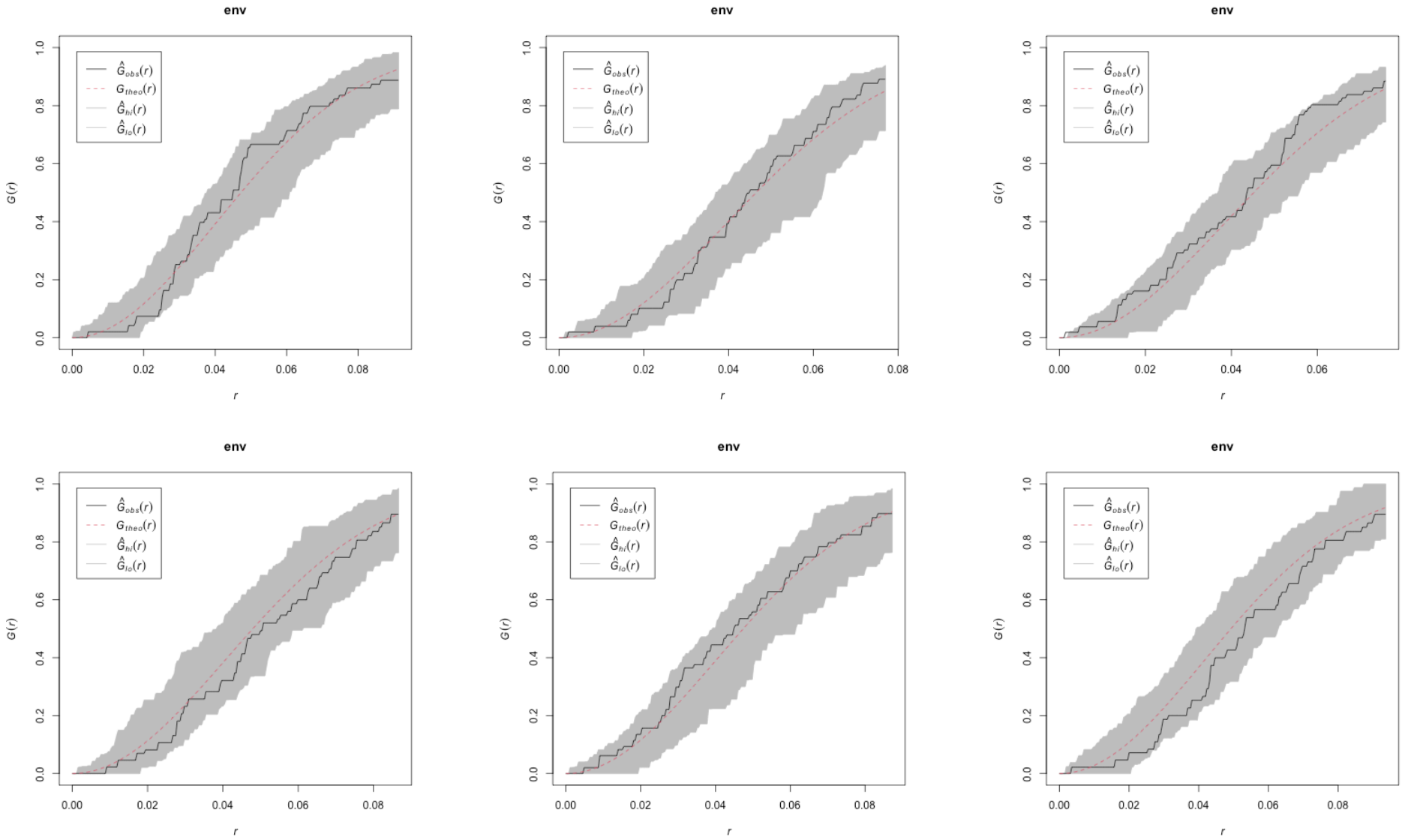
1. compute overall intensity λ
2. repeatedly **simulate** a CSR process with this intensity
3. compute G function for each simulated process
4. the observed pattern is assumed to be a **single realization** of the process
5. realized G function inside the *envelope* $\rightarrow$ evidence that the null hypothesis of CSR can not be rejected

Example envelope

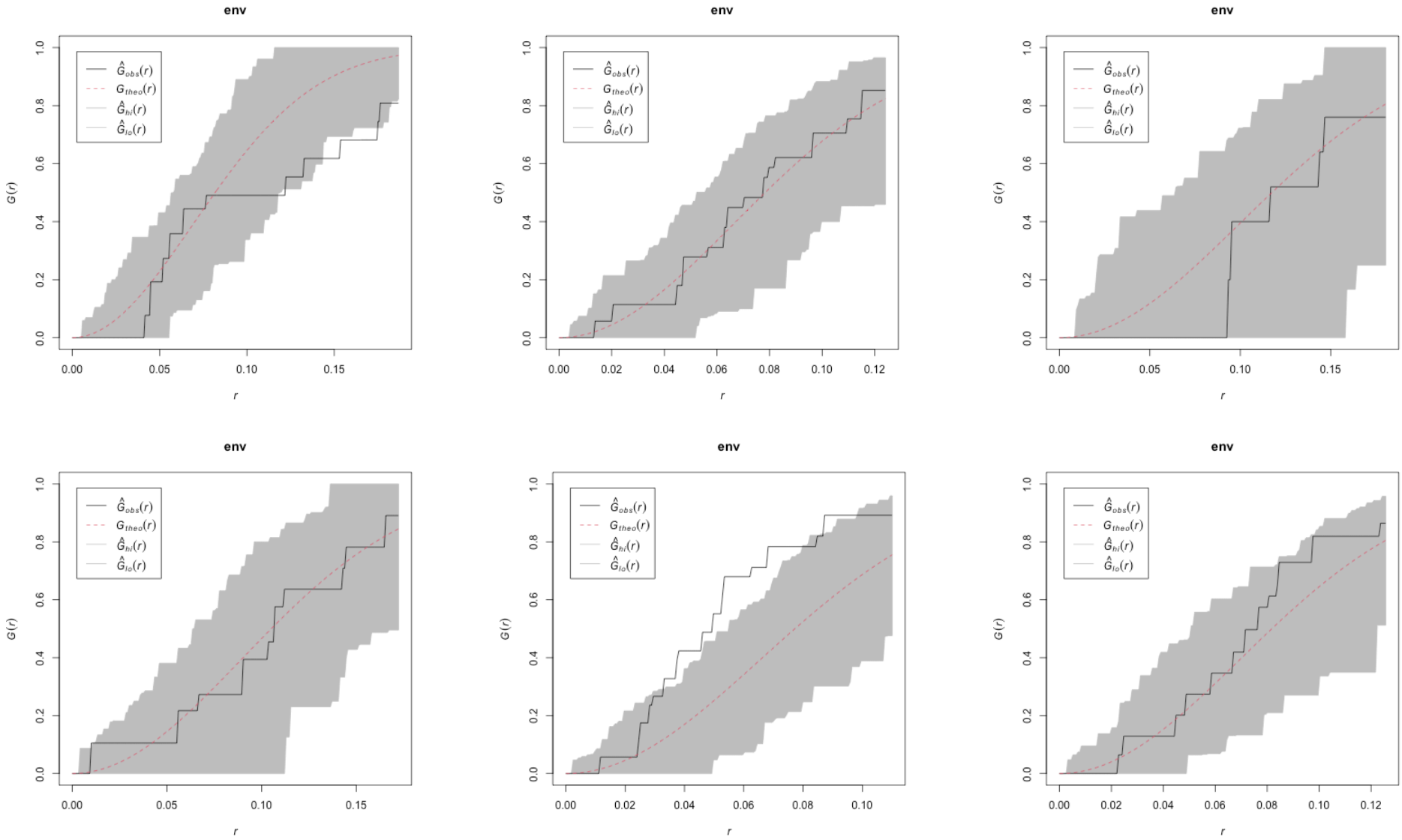


Conclusion: can not reject the null hypothesis of CSR

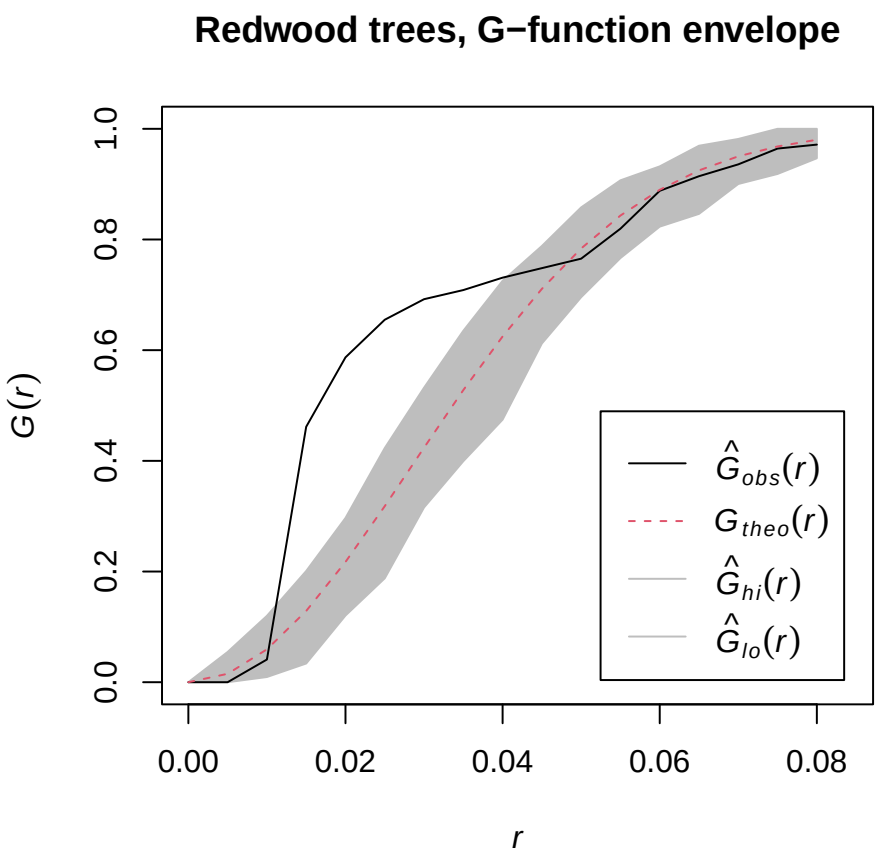
G-function envelope: 100 random points in a unit square

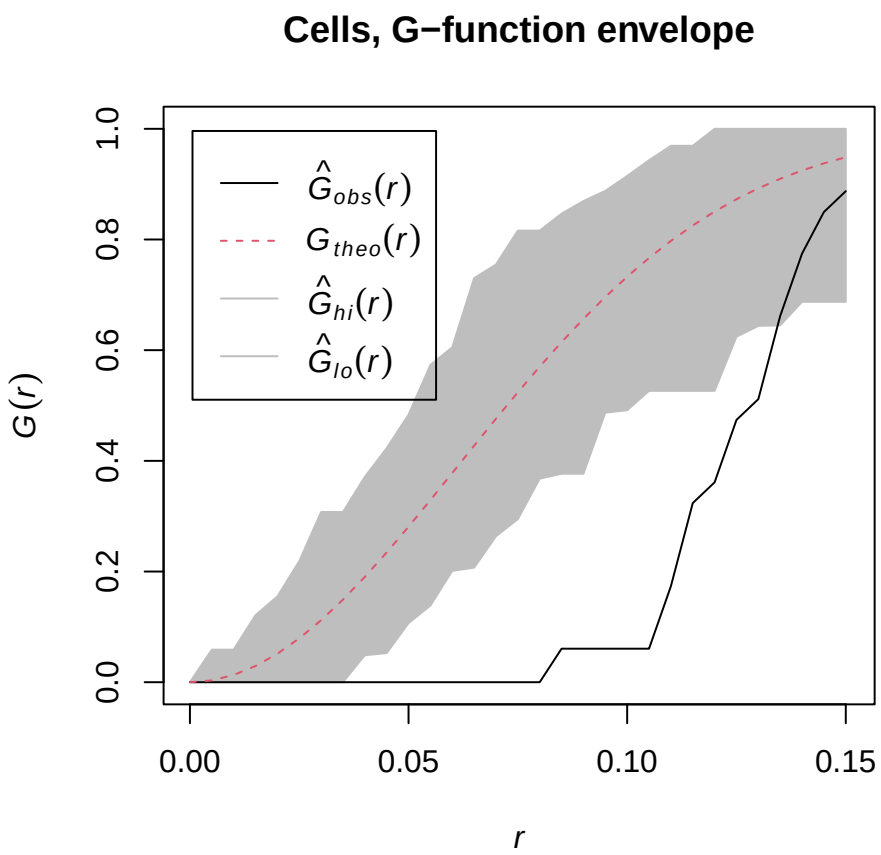


G-function envelope: 30 random points in a unit square



Examples of envelopes for clustered and dispersed patterns





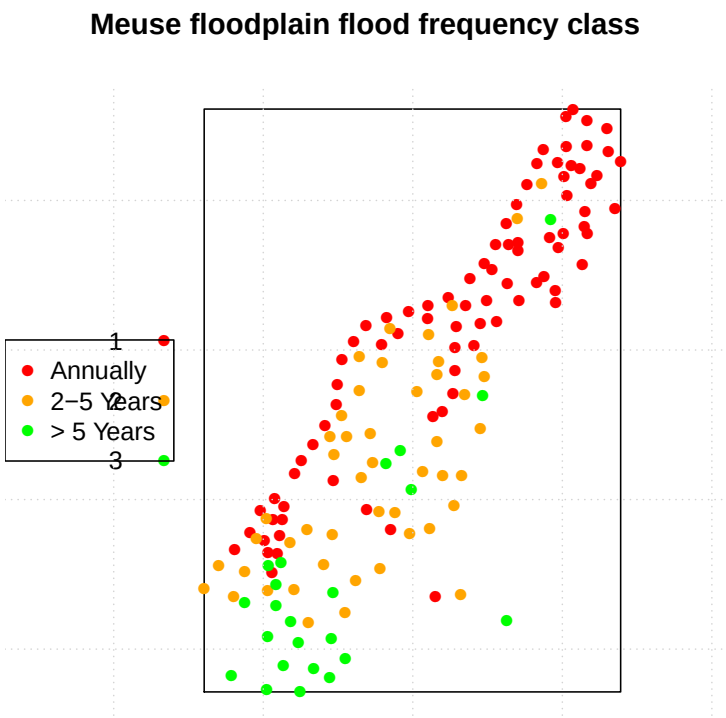
clustered: $\widehat{G}(r) > G(r)$

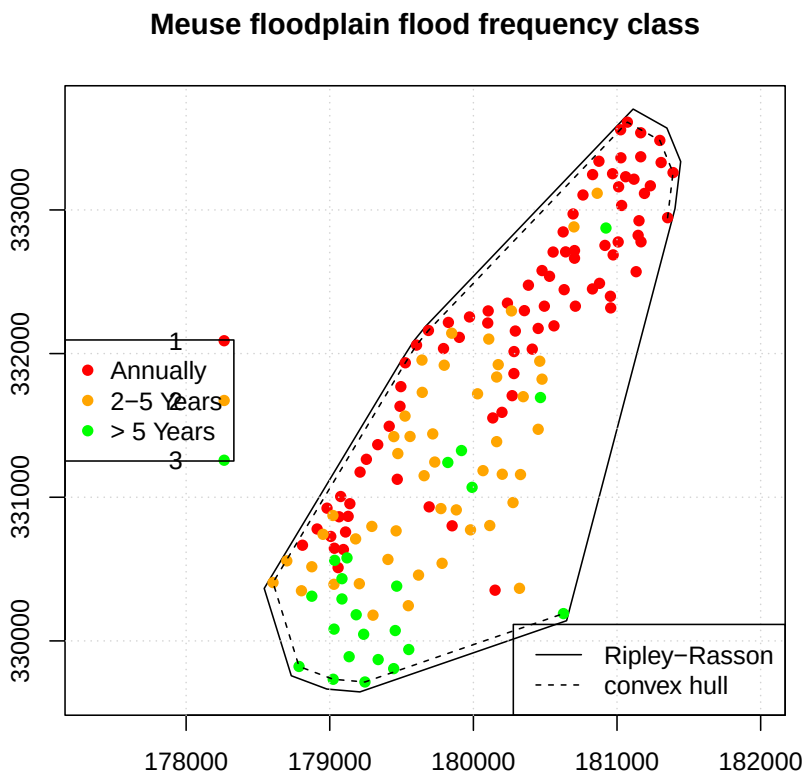
dispersed: $\widehat{G}(r) < G(r)$

The F function

- another way to examine first-order properties
- also called the **empty space** function
- distribution of the distances from an arbitrary **location** (not necessarily a point) to its nearest **observed point**
  - measures the average empty space between observed points.
- it has the same theoretical distribution as the G function
- **sensitive to window size** if there is "empty" space at edges

Effect of window size on F function: windows

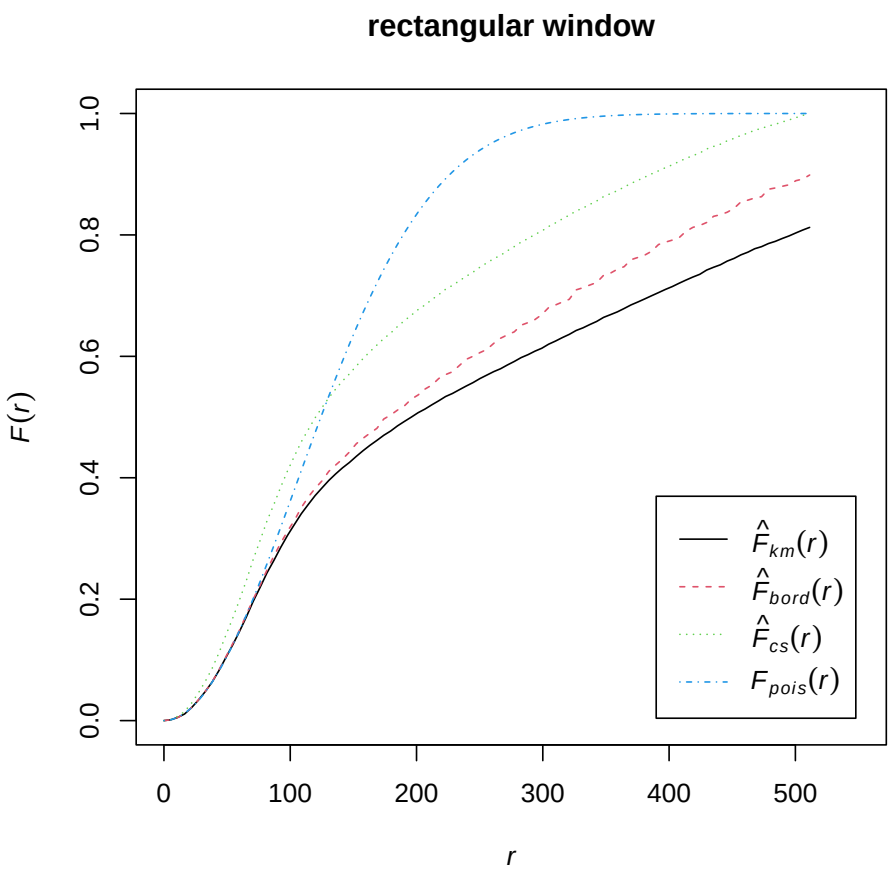


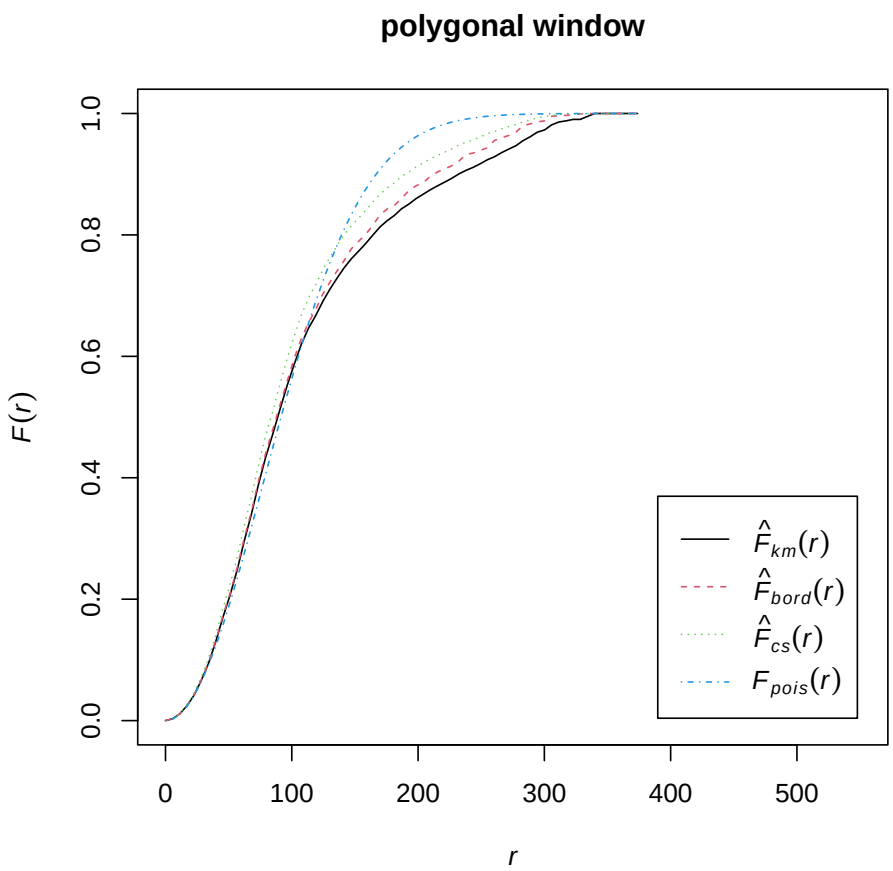


Rectangular

limiting polygon

Effect of window size on F function: results





polygon: much higher proportion of nearest points found within given r than for rectangular bounding box

The J function

- Combines G (point-to-point) and F (space-to-point)
- $J(r) = (1 - G(r))/(1 - F(r))$
- Expected value under CSR $= 1$
  - because G and F have the same expectation under CSR
- $J(r) < 1$ implies clustering, $J(r) > 1$ implies dispersion
- advantage: not sensitive to edge effects

Outline

Second-order properties

- these measure the **interactions** between "events" of one or more sDGP
  - the "events" result in observed points of one or two types
- **competition** (dispersal) within or between processes
  - two trees can not occupy the same position (within some radius)
  - allelopathy: chemicals from one species spread out to some radius, prevent others from growing
- **synergy** (clustering) within or between processes
  - seedlings from one tree sprout nearby? seed dispersal, soil type ...
  - earthquakes facilitated by fracking wells?

Ripley's K function

- measures the **number** of events (points) up to a given distance from an event (point)
  - so, counts **all neighbours** up to that distance
- if $E[.]$ is expectation, $N_0(s)$ is the number of events up to distance s from any event, λ is process intensity:

$$\widehat{K}(s) = \lambda^{-1} E[N_0(s)]$$

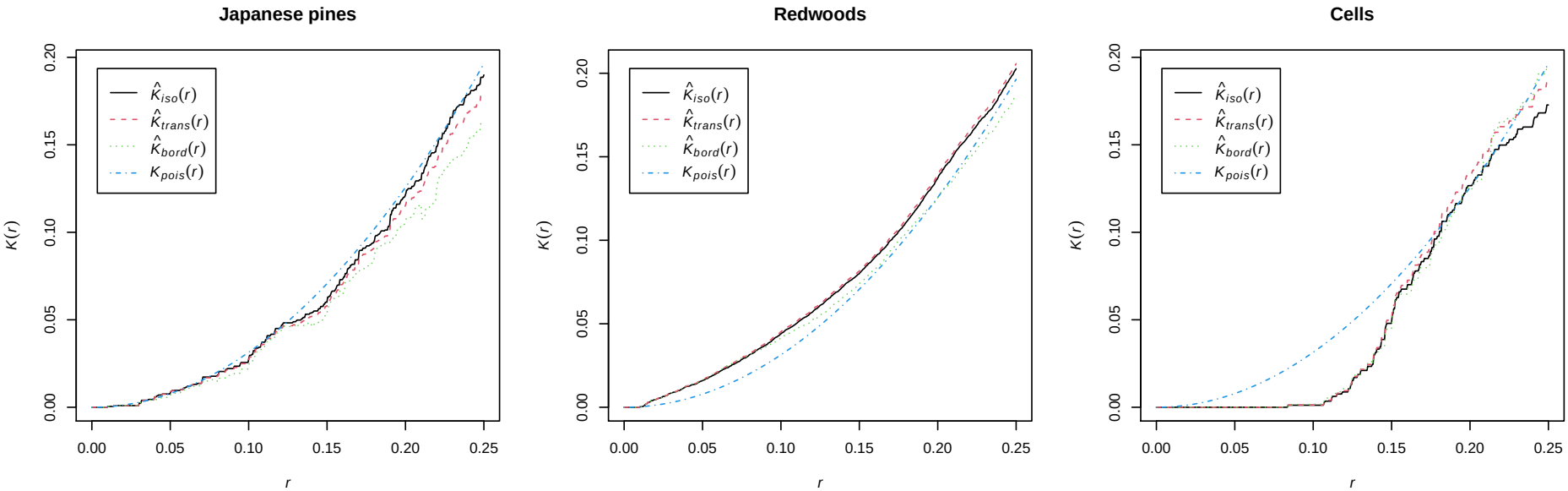
Computing the univariate K function

- unbiased estimator:

$$\widehat{K}(s) = (n(n-1))^{-1}|A| \sum_{i=1}^{2} \sum_{j \neq i} w_{ij}^{-1} |\{x_j : d(x_i, x_j) \leq s\}|$$

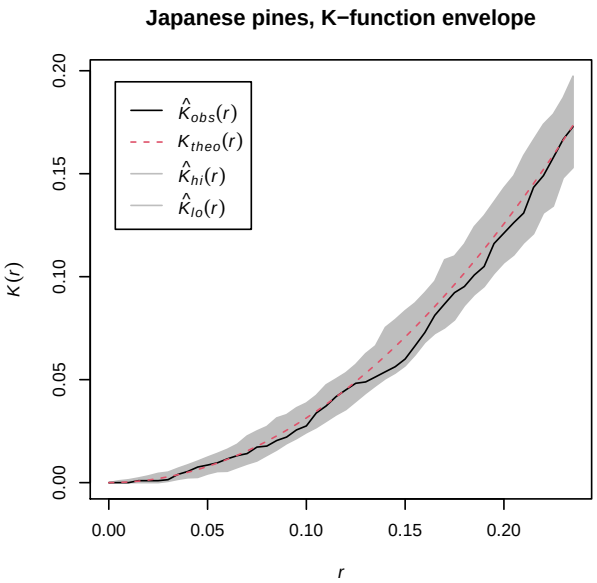
- weights w_{ij}: proportion of $|A|$ occupied by circle centred at x_i with radius $d(x_i, x_j)$
  - corrects for edge effects
- for a homogeneous process, $K(s) = \pi s^2$
  - i.e., number is proportional to circle **area**
- application: graph $\widehat{K}$ and K vs. radius s, compare actual to theoretical at each s
- can compute envelopes as for G and F functions

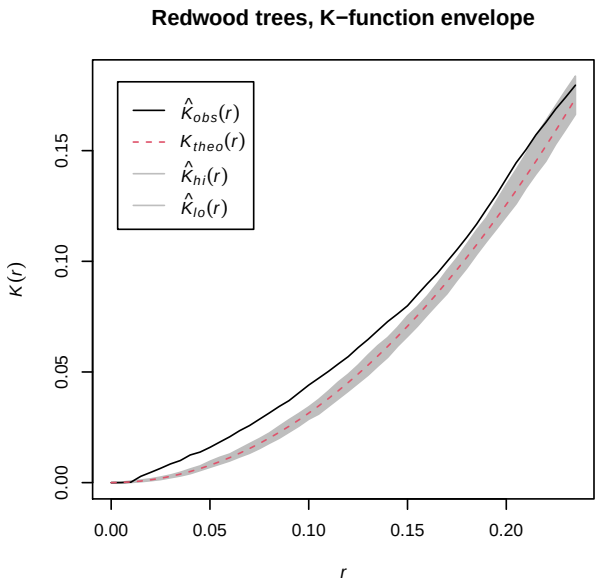
K function results

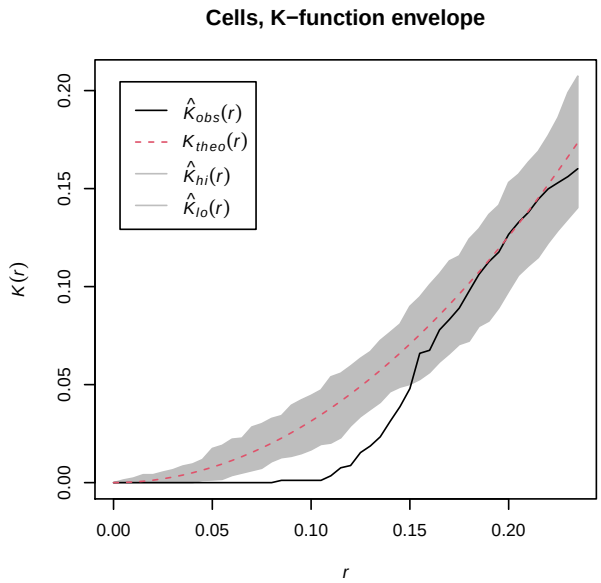


- *Japanese pines*: empirical matches the theoretical for no relation;
- *redwoods*: more than expected events near to an arbitrary event; "attraction" $\rightarrow$ clustering
- *cells*: fewer nearby events, after 0.15 units there is no more "repulsion" $\rightarrow$ dispersal
- **interpret** in terms of sDGP (note the radius)

K function envelopes



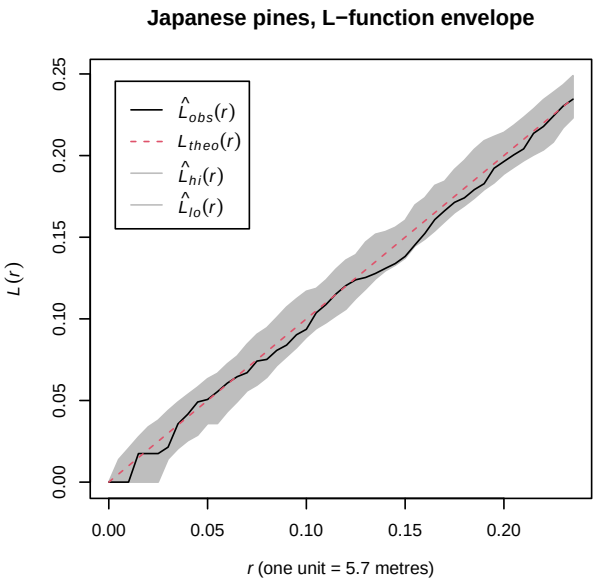


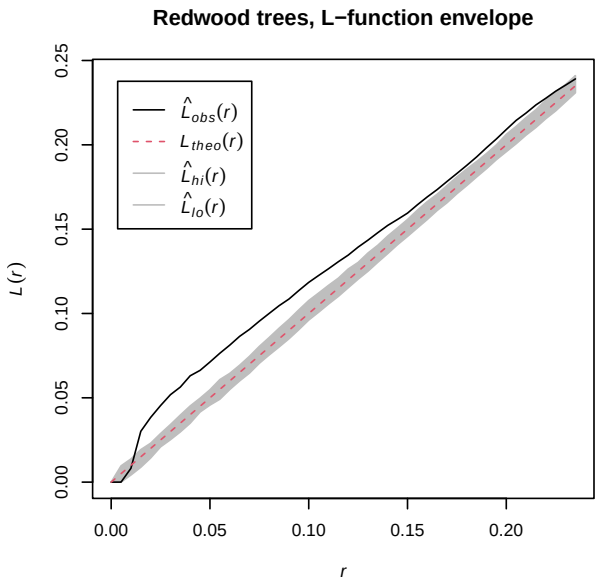


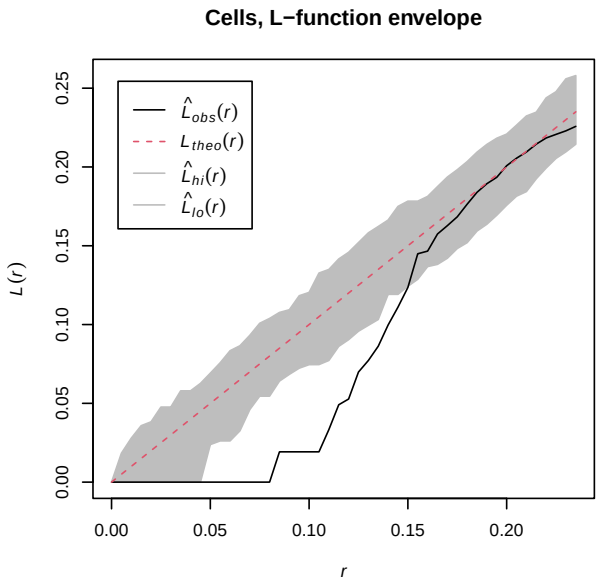
Besag's L function

This is a linearization of the K-function which makes it easier to compare theoretical and actual values at narrow separations (low values of the radius):

$$L(r) = \sqrt{\frac{K(r)}{\pi}}$$







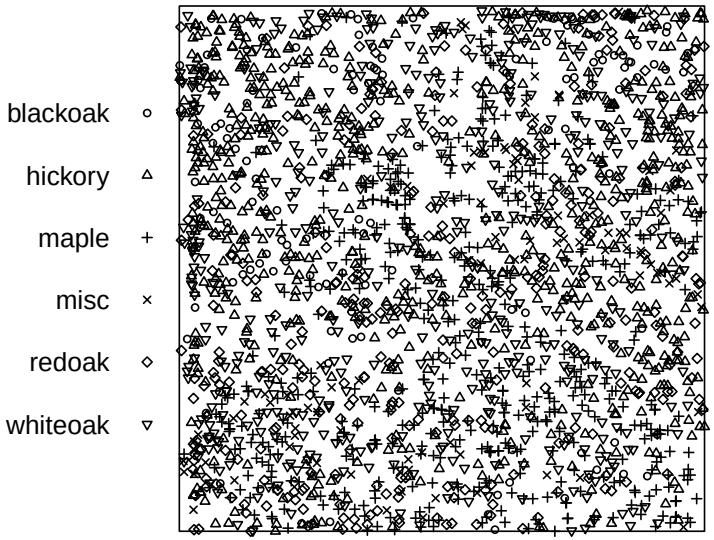
Outline

Marked point patterns

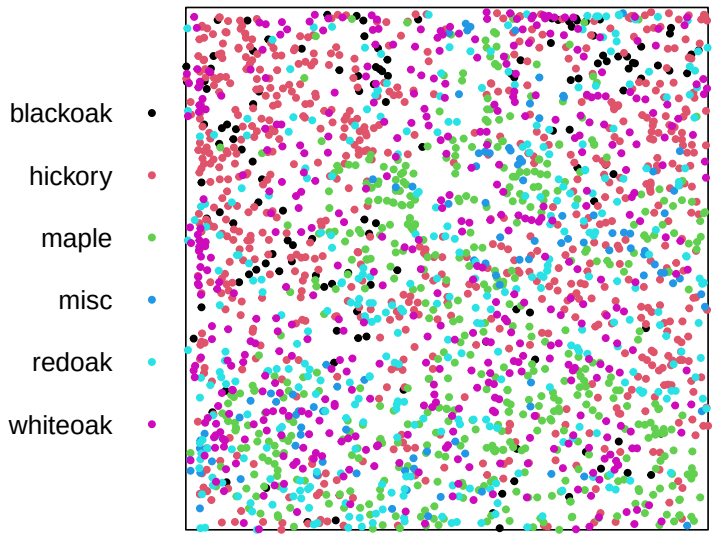
- The previous measures were just considering point **location**, with no information about what the point represents.
- But a point pattern can be **marked**: each point is **labelled** with some attribute
  - Example: tree species in a forest plot (**categorical** mark)
  - Example: size of trees in a forest plot (**continuous** mark)
  - Example: **time** of observation of a point (e.g., wildfires)
- Can analyze each sub-pattern separately (as with unmarked patterns)
- Can analyze **interactions** between patterns

Example marked pattern

Lansing woods, species shown by symbol

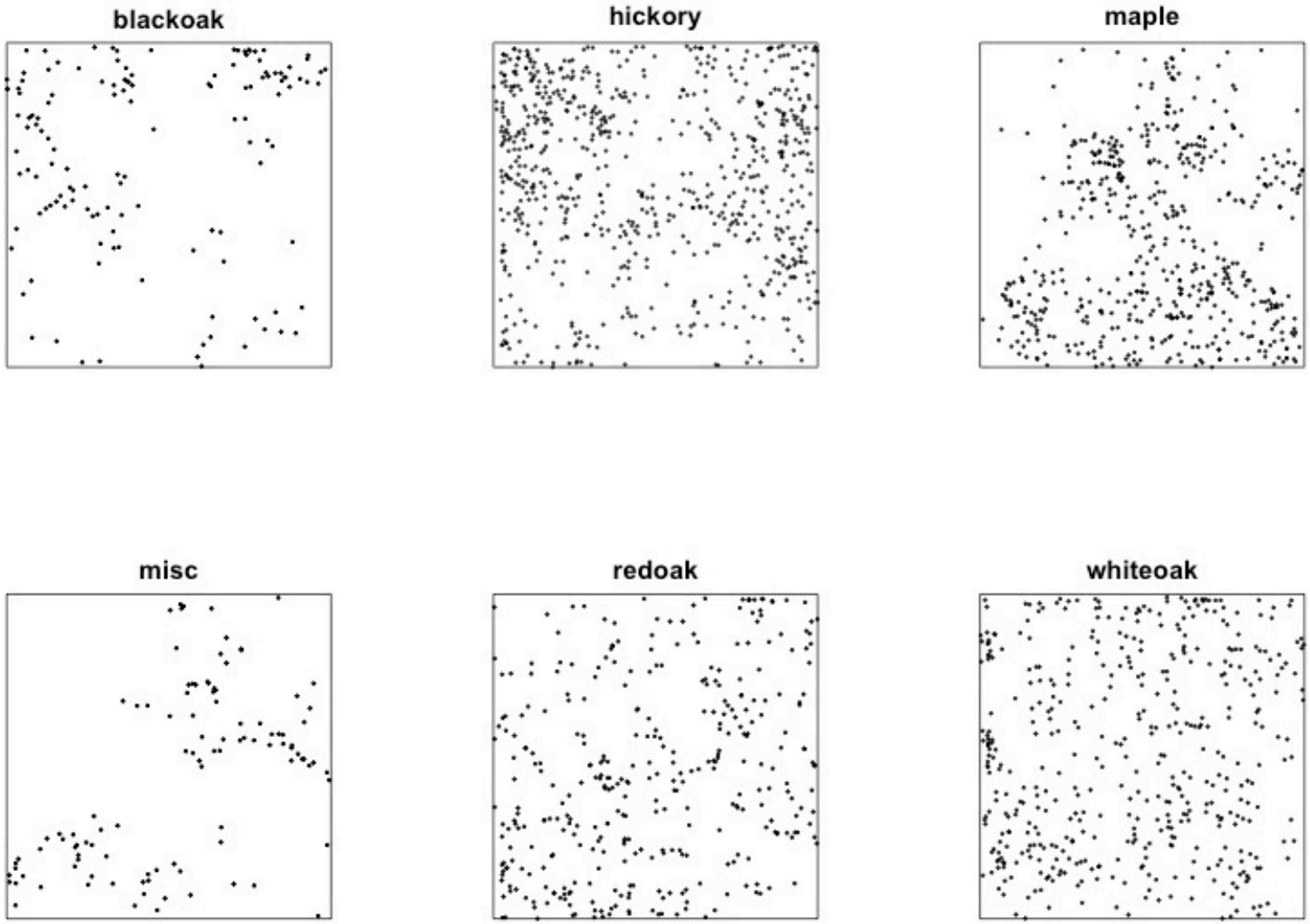


Lansing woods, species shown by colour



Location of trees in Lansing woods, **marked** by their **species**

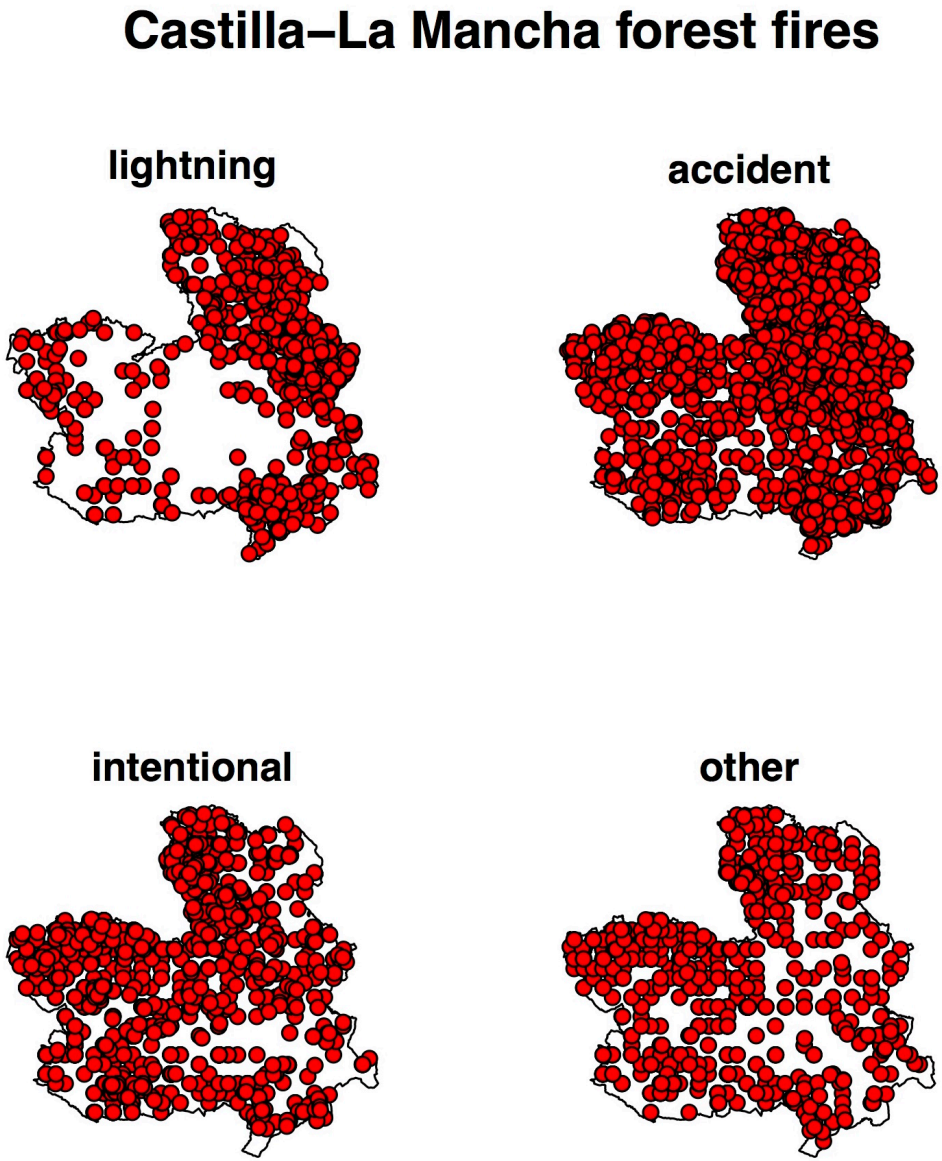
Marked pattern divided into unmarked patterns



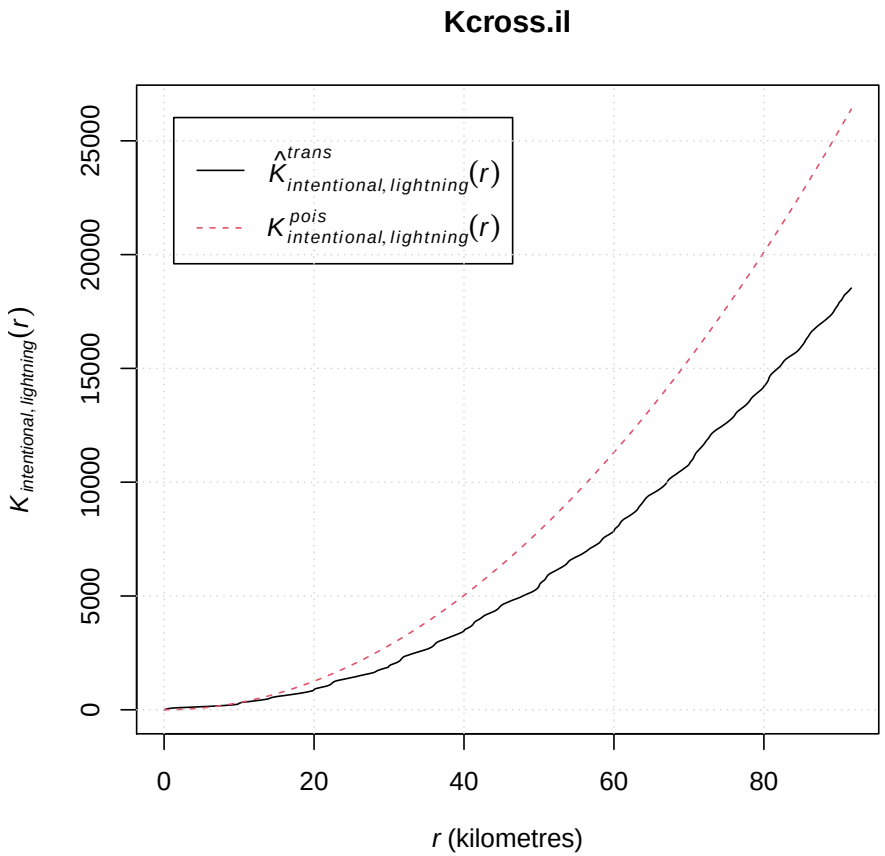
Interactions in marked patterns

- Q: Do patterns with different marks "influence" each other?
  - Simpler: what is their relation?
- A: Bi- and multi-variate versions of G, K, L, J functions (not F)
  - $G_{ij}(r)$: the distribution of the distance from a typical point of type (mark) i to its nearest point of type j.
  - $K_{ij}(r)$: given intensity λ_j of type j, $\lambda_j K_{ij}(r)$ is the expected number of *additional* points of type j within a distance r of a typical point of type i.
  - empirical $>$ theoretical $\rightarrow$ clustering & vice-versa.

Example marked pattern: forest fire type

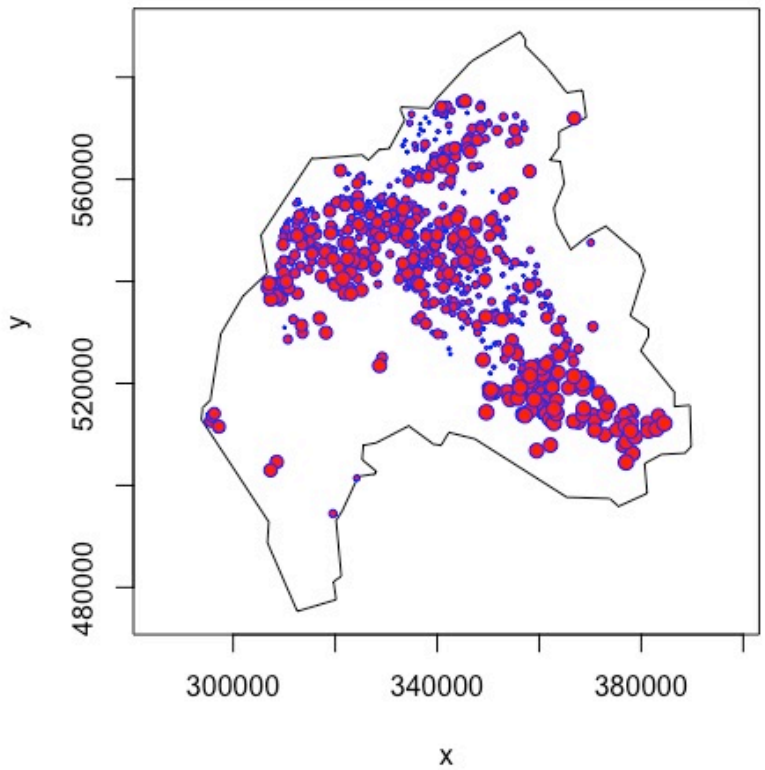


Example crossed-K function

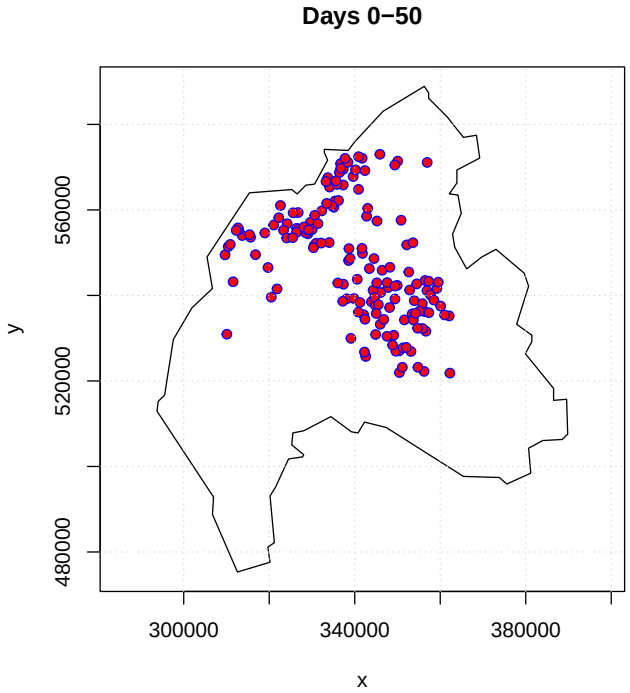


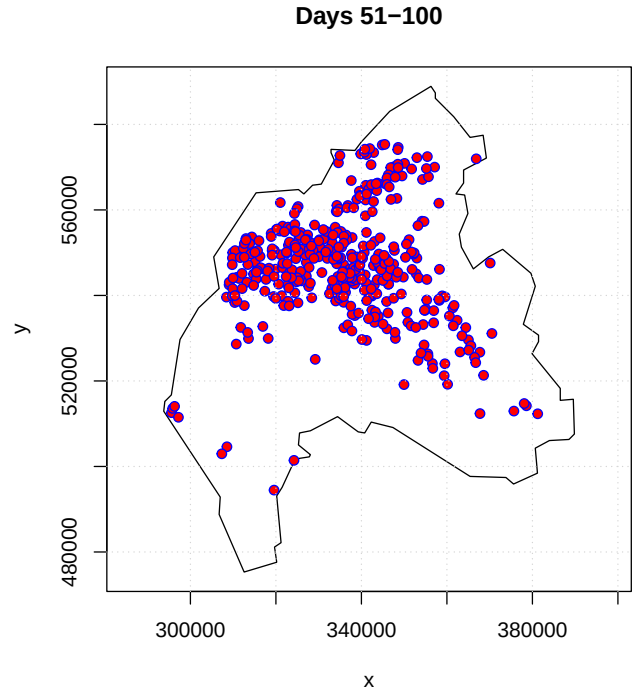
Intentional vs. caused by lightning
After 15 km radius there is “repulsion”

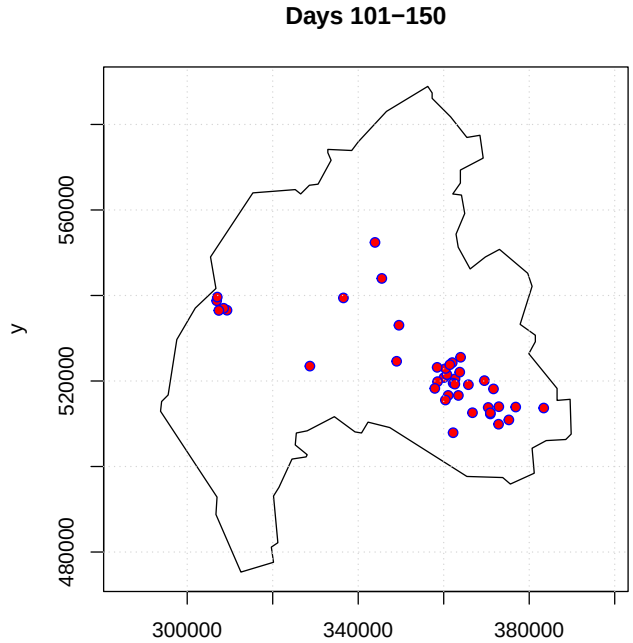
Marked pattern: time of occurrence

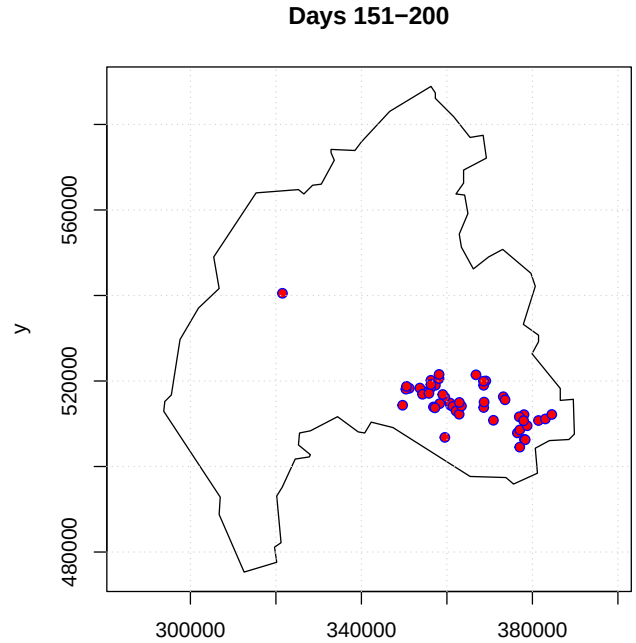


Foot-and-mouth disease, northern Cumbia (England), 2001; from R package `stpp`, dataset `fmd`; more recent $\rightarrow$ larger symbol

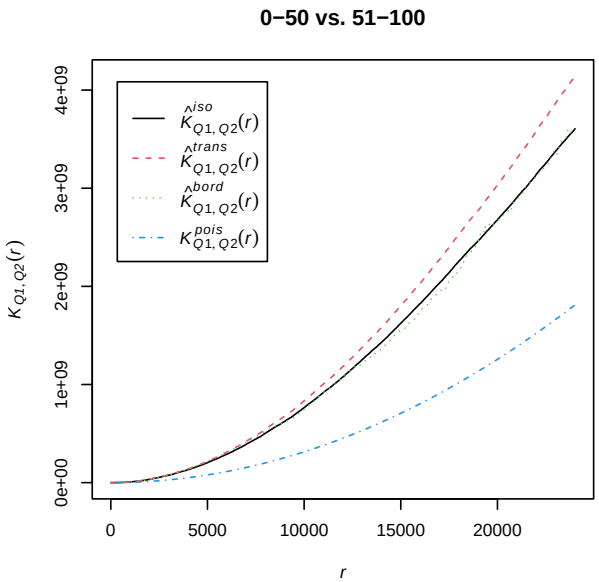

Days 0−50
y
560000
520000
480000
300000
340000
380000
x

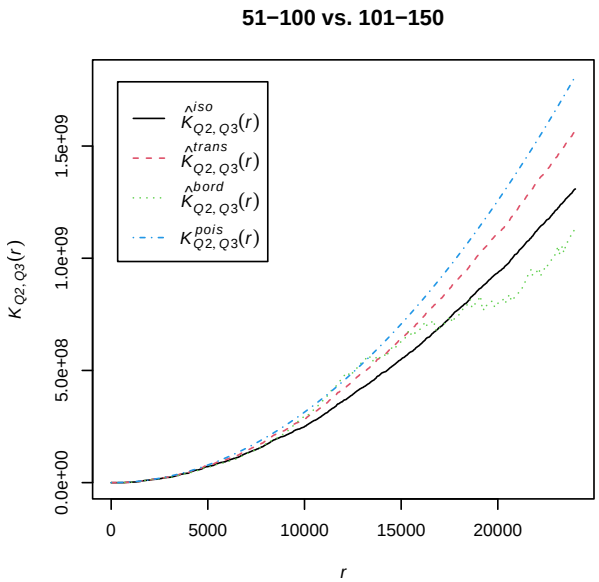

Days 51−100
y
560000
520000
480000
300000
340000
380000
x

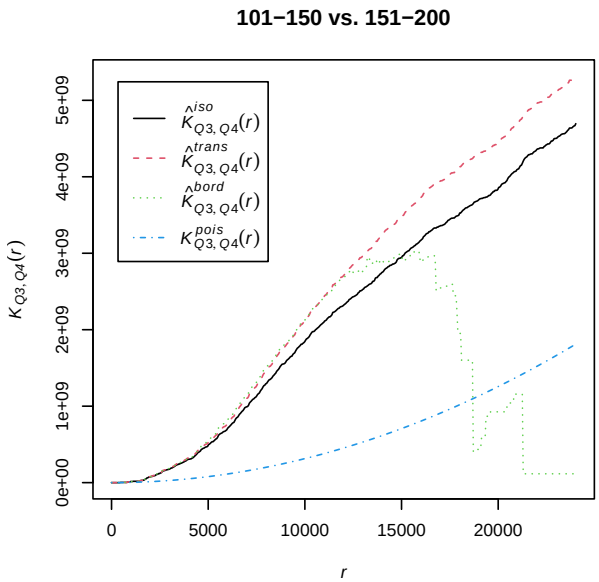

Days 101−150
y
560000
520000
480000
300000
340000
380000


Days 151−200
y
560000
520000
480000
300000
340000
380000

Crossed K-function







Simple spatio-temporal analysis

- Marked point-pattern, marks as time of occurrence
- Use the crossed K etc. functions to assess interaction

Outline

Models of spatial data-generating processes

- We often want to infer the sDGP that produced the observed pattern
- We have competing **models**
  - CSR, attraction, repulsion
  - dependence on spatially-distributed **covariables**
  - interaction with other sDGP

Model development and evaluation

- formulate hypothetical model
  - based on *a priori* knowledge or theory
  - example from V-1 rockets: no in-flight guidance system
  - example from ecology: allelopathy
  - example from criminology: distance from source, "no action" buffer
- parameterize with observed pattern(s) and possibly covariables
- evaluate **goodness-of-fit**
  - good fit $\rightarrow$ **evidence** (*not* "proof"!) for the hypothesized sDGP

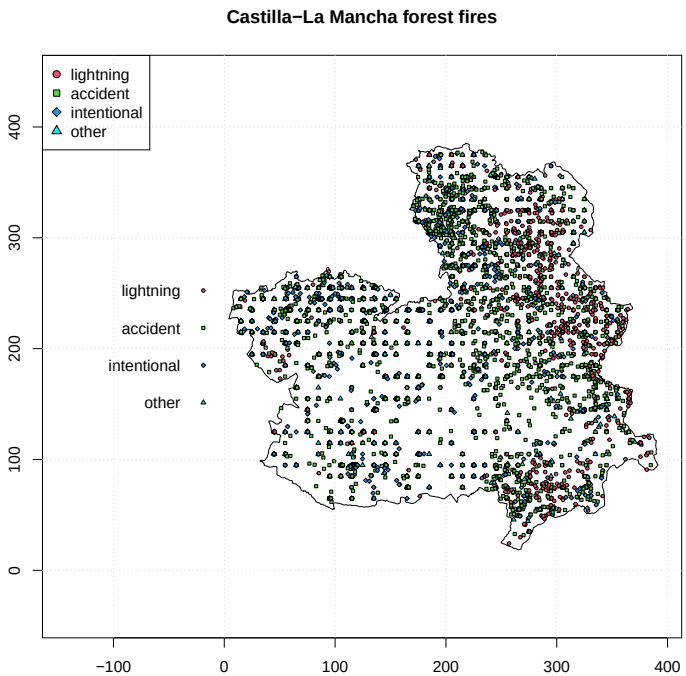
A general model of sDGP

- spatial **trend** – a function of the coördinates
- **interaction** between events
- influence of **covariates** (other than trend) on events
- General model:

$$\lambda(s, \mathbf{x}) = \exp(\psi^T B(s) + \phi^T C(s, \mathbf{x}))$$

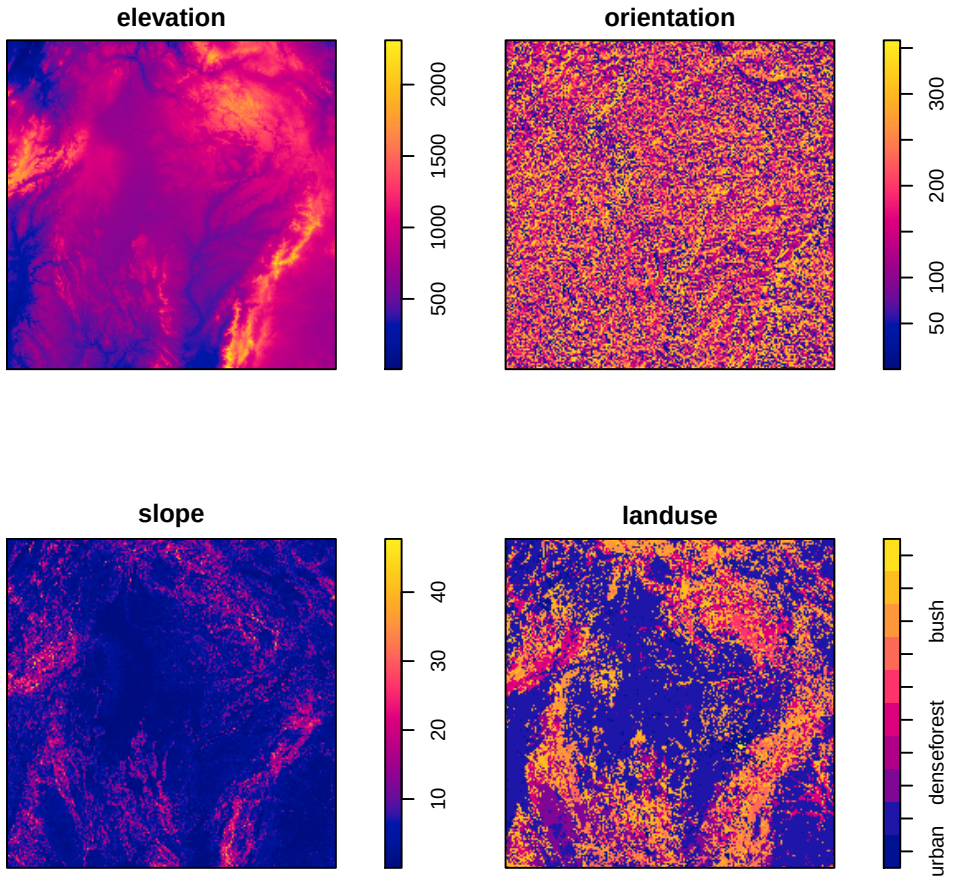
  - $B(s)$: depends only on location (trend and/or spatial covariates)
  - $C(s, \mathbf{x})$ also depends on other points
  - either may be absent (simpler process)
  - details in Baddeley and Turner [1]
- compute with `spatstat` package function `ppm`

Example sDGP covariates – Spanish fires



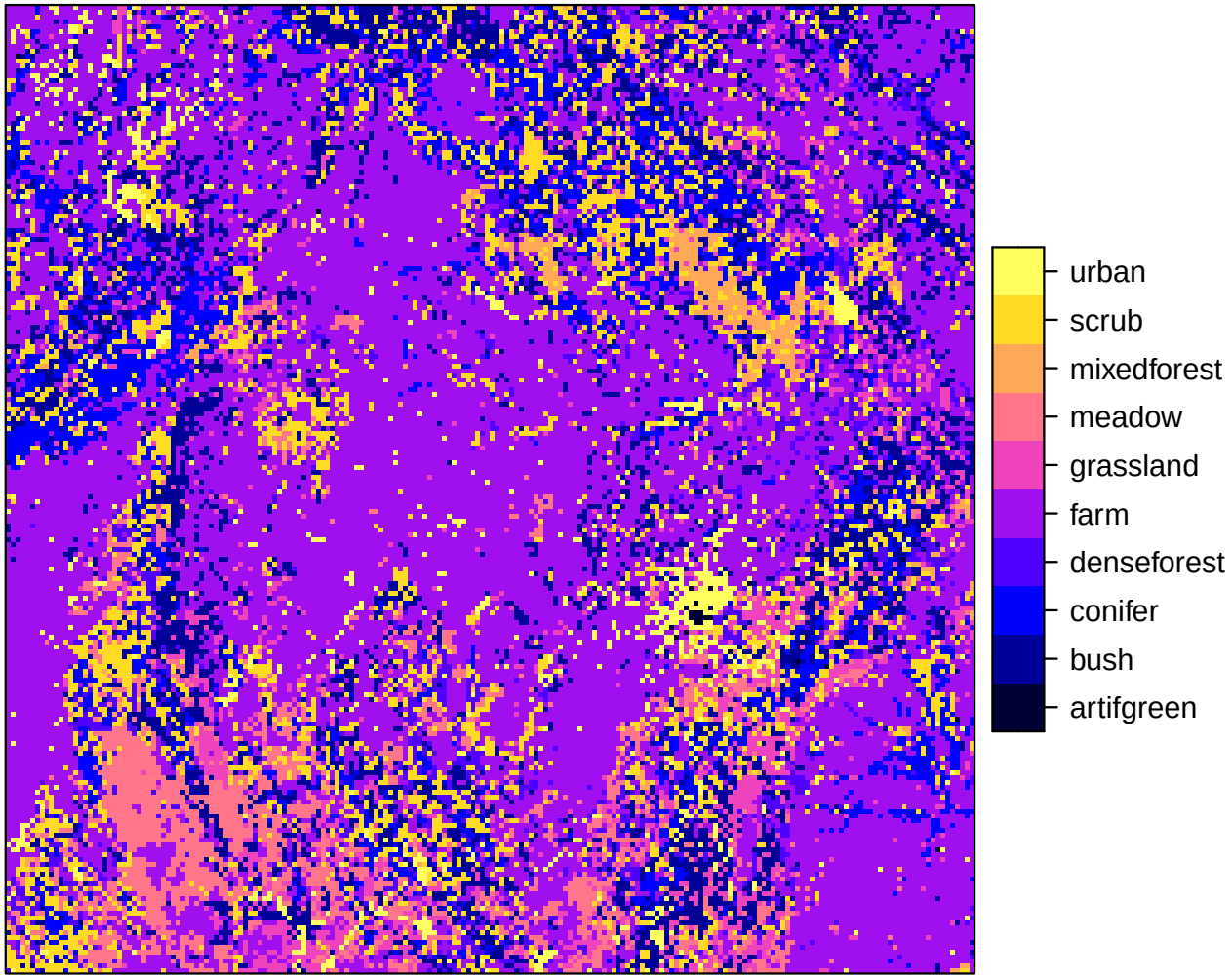
Pattern of fires

200 m grid covariates



spatial covariates

sDGP covariate – land cover classes



sDGP models of Spanish fires to compare

1. **Null**: complete spatial randomness (CSR)
   - homogeneous Poisson process
2. **Trend**: regional trend in the coördinates in conditional density
   - **in**homogeneous Poisson process
   - may be due to a trend in an environmental factor, e.g., rainfall
3. **Covariates**: dependence on mapped covariates
   - e.g., some land uses more prone to fire
4. **Interaction**: clustering or repulsion between points: **Strauss** processes
5. **Combination** of trend and/or covariates and/or interaction

Model selection

- Based on reasonable **hypotheses** of the sDGP
- Evaluate by statistical **likelihood** that the observed pattern was generated by the hypothesized sDGP
- Compare **alternate models** by **relative** likelihood
- Interpret **parameters**: strength of inter-point interaction; coefficients of trend surfaces or covariate models; relative strength of trend and interaction components
  - Do we now understand the sDGP better? Are our hypotheses confirmed, rejected, modified?

Comparing models by likelihood – Spanish fires

1. **Poisson** model: CSR
2. **Landuse** model: observed density depends only on landuse
3. **Strauss process** model: observed density depends only on interaction between events
4. Landuse + Strauss model: **combined**

```
              model likelihood
1           Poisson    -8562.0
2           Landuse    -8532.2
3           Strauss    -6749.9
4 Landuse + Strauss    -6730.8
```

Landuse explains little; interaction explains a lot; combined is best

Interpreting model parameters – Spanish fires

Interaction: $\gamma > 1$ suggests clustering; β is overall log-density

```
> exp(coef(m.strauss.4))
(Intercept) Interaction
   0.017543    1.080151
```

Land use:

```
Fitted coefficients for trend formula:
       (Intercept)        landusefarm      landuseconifer
          -4.41477            0.28745             0.59341
```

Higher density in coniferous forest than on farms

Trend surface model – Spanish fires

```
> (m.ts1 <- ppm(clmfires.i, trend = ~polynom(x, y, 1),
     interaction = NULL))

Nonstationary Poisson process

Log intensity:  ~x + y

Fitted trend coefficients:
(Intercept)           x           y
 -3.2878809  -0.0057327   0.0028847

              Estimate       S.E.    CI95.lo    CI95.hi Ztest    Zval
(Intercept) -3.2878809 0.07317041 -3.4312923 -3.1444695   *** -44.935
x           -0.0057327 0.00027073 -0.0062633 -0.0052021   *** -21.175
y            0.0028847 0.00030260  0.0022916  0.0034778   ***   9.533
```

Comparing trend surface models by ANOVA – Spanish fires

1. **Poisson** model: CSR
2. **First-order trend** in the coördinates
3. **Second-order trend** in the coördinates

```
> anova(m.ts2, m.ts1, m.pois)

Analysis of Deviance Table

Model 1: ~x + y + I(x^2) + I(x * y) + I(y^2) ^^I Poisson
Model 2: ~x + y ^^I Poisson
Model 3: ~1 ^^I Poisson
  Npar Df Deviance
1    6
2    3 -3     -126
3    1 -2     -505
```

Second-order model is a bit better than the others.

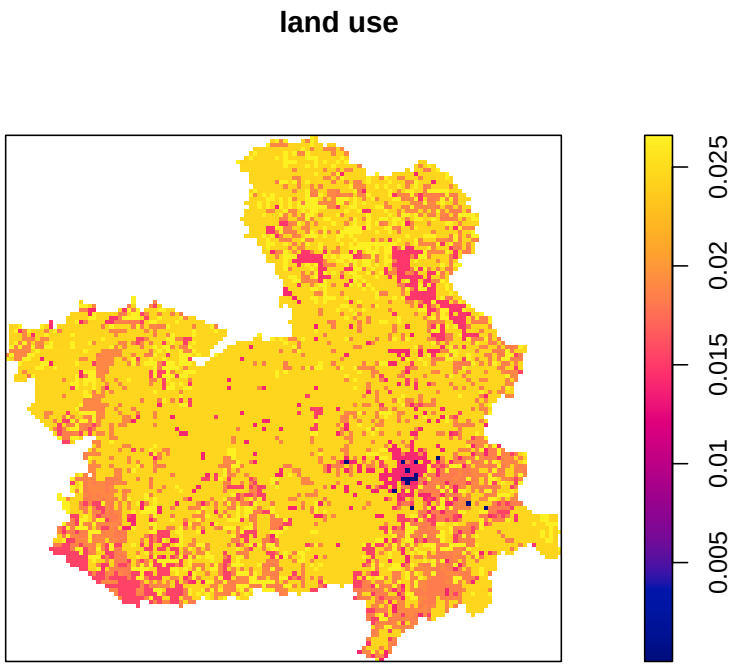
Prediction from sDGP models

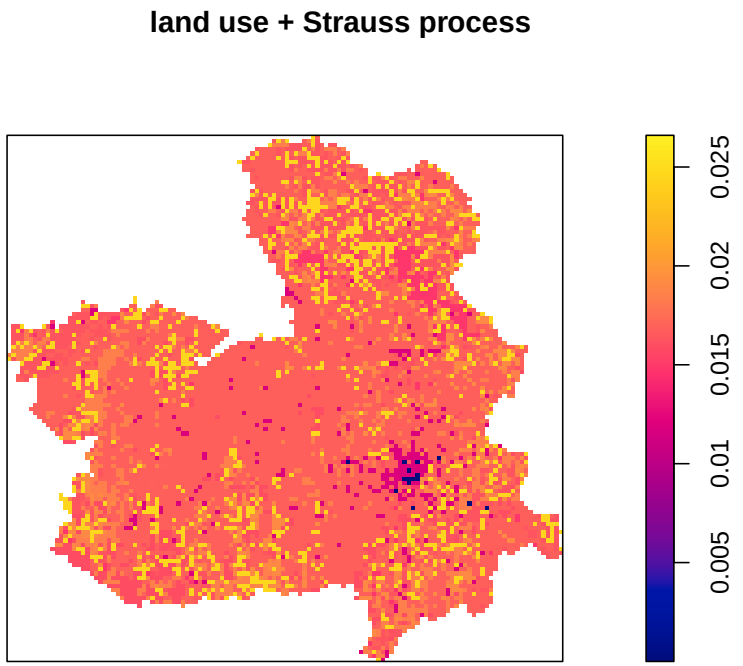
- Once an sDGP model is **fit**, it has **parameters**
  - e.g., coefficients of a trend surface or covariate model, strength of inter-point interaction.
- This model can be applied to new situations: across an area (from trend, points), with renewed covariates
- `spatstat` function `predict.ppm` predicts from models fit with `ppm`.

Example predictions – Spanish fires

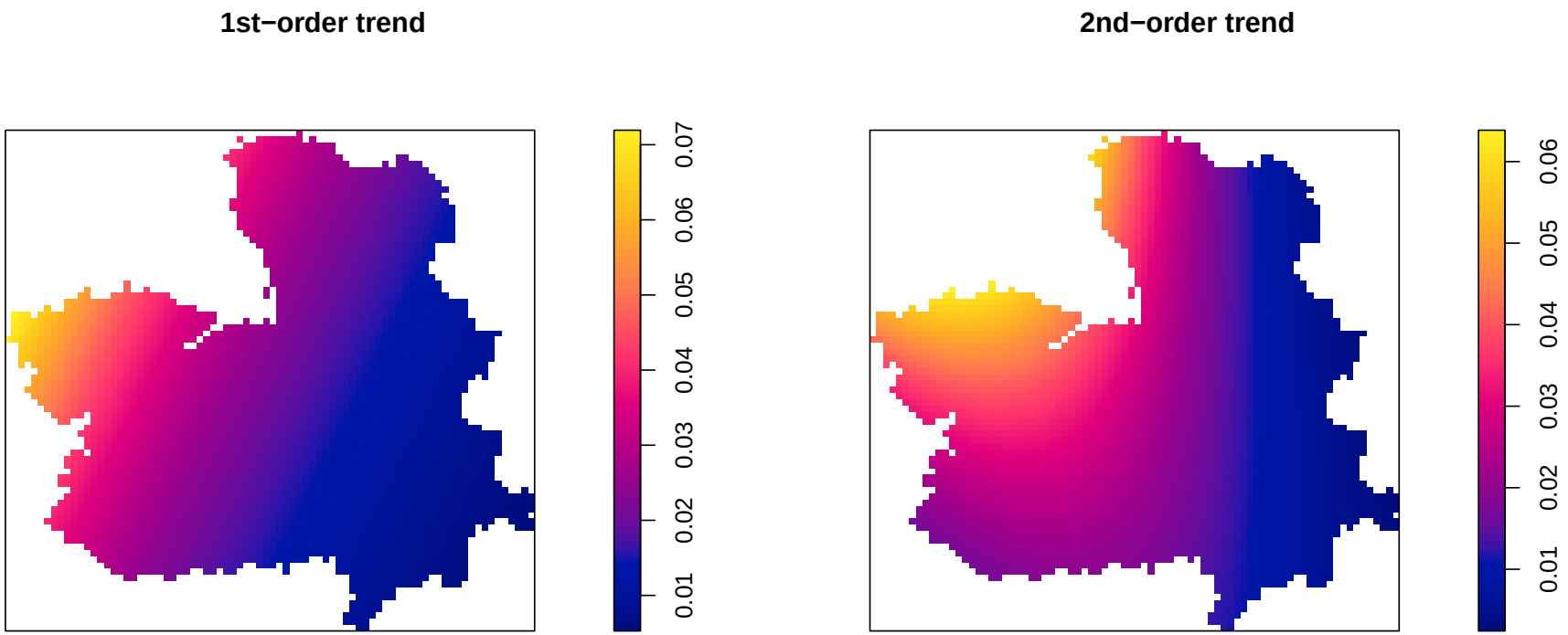
```
> pred.lu.strauss <- predict(m.lu.strauss.4,
                             covariates=clmfires.extra$clmcov200)
> summary(pred.lu.strauss)

real-valued pixel image
128 x 128 pixel array (ny, nx)
enclosing rectangle: [4.1311, 391.38] x [18.565, 385.19] kilometres
dimensions of each pixel: 3.03 x 2.8642 kilometres
Image is defined on a subset of the rectangular grid
Subset area = 79462.0730449286 square kilometres
Subset area fraction = 0.56
Pixel values (inside window):
^^Irange = [2.2603e-07, 0.024494]
^^Iintegral = 1402.8
^^Imean = 0.017653
```

Model predictions – covariate, Strauss process





Model predictions – trend surface

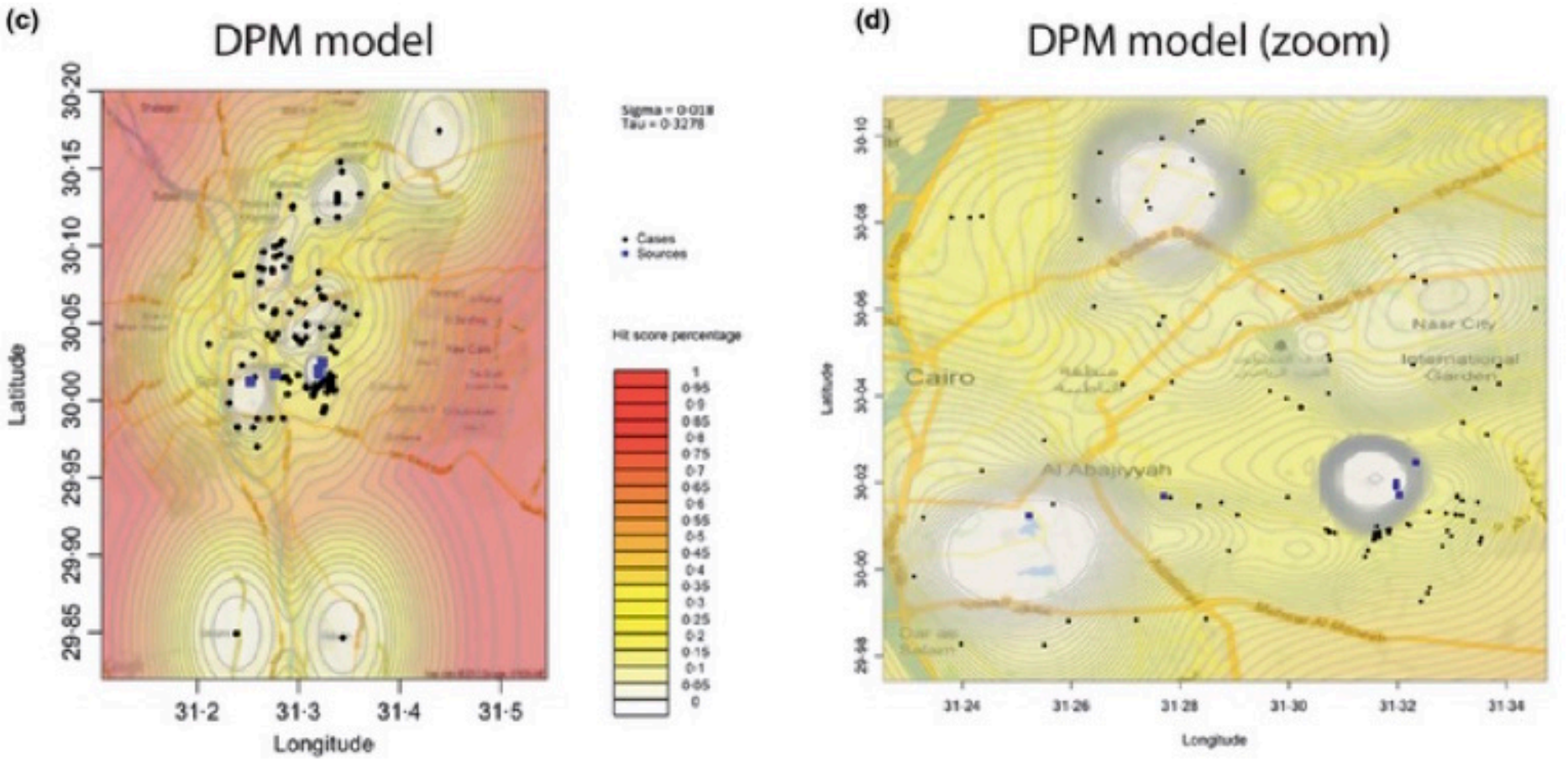


Outline

Empirical source finding

- **Objective**: find **possible sources** from a set of **occurrences**.
- **Approach**: empirical criminal geographic targeting (CGT)
- **Approach**: Dirichlet Process Mixture (DPM)
  - no predefined number of clusters; algorithm finds most probable and their location

Example of source finding



139 *Plasmodium vivax* cases in Cairo, Egypt
observed data points; black circles; empirically-identified **sources**:blue squares.

source: Verity et al. [18]

Bayesian models

- Explicit probability models
- Can incorporate knowledge via **prior probabilities**
  - distribution and parameters, e.g., normal with a prior mean and variance
  - e.g., weights of passengers and luggage on a flight
- **Update** probabilities based on **evidence**
  - for realistic models computationally-intensive (e.g., Markov chain Monte Carlo)
- Result is a **posterior probability** of parameters of the chosen distribution

Bayesian Hierarchical Models

- The model of the effect (e.g., spatial pattern of a disease) is specified as a **hierarchical** set of layers
- Each layer accounts for different sources of spatial variation
- E.g., Besag et al. [2]: sum of:
  1. a **spatially-correlated** variable;
  2. an **area-independent** effect (local heterogeneity)

INLA

- **"Integrated Nested Laplace Approximation"** to the posterior marginals of model parameters
- INLA computes only **relative** posterior distributions for *latent Gaussian models*
  - these are enough for many applications, e.g., relative risks in disease mapping
- References: Rue et al. [15], Illian et al. [11]; Bivand et al. [4]; `http://www.r-inla.org`

Outline

Further reading

Theory: Boots and Getis [5], Diggle [7], Ripley [14], Matérn [12], Illian et al. [10] (also as e-book)

Applications: Gatrell et al. [9]

In R: Bivand et al. [3, Ch. 7] (also as e-book); Baddeley and Turner [1]
Spatial simulations of various processes (Paul Keeler):
`https://github.com/hpaulkeeler/posts/tree/master` from the blog
`https://hpaulkeeler.com/posts/`

Web pages

- Manuel Gimond (Colby College): `https://mgimond.github.io/Spatial/chp11_0.html`
- Roger Bivand (Norwegian School of Economics, Bergen): `https://rsbivand.github.io/ECS530_h21/ECS530_211119.html#Point_patterns`

R packages

spatial	Functions for Kriging and Point Pattern Analysis (Ripley, Bivand, Venables)
spatstat	Spatial Point Pattern Analysis, Model-Fitting, Simulation, Tests (Baddely, Turner *et al.*)
splancs	Spatial and Space-Time Point Pattern Analysis, within a polygonal region of interest (Bivand, Rowlingson, Diggle *et al.*)
Rgeoprofile	Geographic profiling in R (Stevenson, Verity, Nichols, LeComber)
R-INLA	Integrated Nested Laplace Approximation (INLA)

See also `https://cran.r-project.org/web/views/Spatial.html`, section "Point pattern analysis"

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