Tutorial: Using the R Environment for Statistical Computing
An example with the Mercer & Hall wheat yield dataset

David G Rossiter

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1 Introduction

This tutorial introduces the **R environment for statistical computing and visualisation** [24, 42] and its dialect of the **S language**. It is organized as a systematic analysis of a simple dataset: the Mercer & Hall wheat yield uniformity trial (Appendix A). After completing the tutorial you should:

- know the basics of the R environment;
- be able to use R at a beginning to intermediate level;
- follow a systematic approach to analyze a simple dataset.

The tutorial is organized as a set of tasks followed by questions to check your understanding; answers are at the end of each section. If you are ambitious, there are also some challenges: tasks and questions with no solution provided, that require the integration of skills learned in the section.

Not every section is of equal importance; you should pick and choose those of interest to you. Sections marked with an asterisk ‘*’ are interesting “detours” or perhaps “scenic byways” developed to satisfy the author’s curiosity or to answer a reader’s question.

R is an open-source environment for data manipulation, statistical analysis, and visualization. There are versions for MS-Windows, Mac OS/X, and various flavours of Unix. It is most convenient to run R within an **integrated development environment** (IDE), e.g., RStudio\(^1\); this environment runs on Microsoft Windows, OS/X and Linux.

**Note:** For an explanation of the R project, including how to obtain and install the software and documentation, see Rossiter [44]. This also contains an extensive discussion of the S language, R graphics, and many statistical methods, as well as a bibliography of texts and references that use R.

2 R basics

Before entering into the sample data analysis (§3), we first explain how to interact with R, and the basics of the S language. The simplest way to interact with the R environment is by typing commands at the “R console” command line; this is one of the windows in the RStudio IDE.

---

**Task 1**: Start R, preferably within an IDE such as RStudio.

If you use RStudio, the screen will something like Figure 1.

After starting R, you will be looking at a console (or, within the IDE, a console window) where you interact with R: giving commands and seeing numerical results; graphs are displayed in their own windows. You perform most actions in R by typing commands in response to a command prompt, which usually looks like this:

\(^1\)http://www.rstudio.org; there is a complete list of code editors and IDE’s at http://www.sciviews.org/_rgui/projects/Editors.html
The > is a prompt symbol displayed by R, not typed by you. This is R’s way of telling you it’s waiting for you to enter a command.

Type your command\(^2\) and press the Enter or Return keys; R will execute (carry out) your command.

Sometimes the command will result in numerical output listed on the console, other times in a graph displayed in a separate window, other times R will just do what you requested without any feedback.

If your entry is not a complete R command, R will prompt you to complete it with the continuation prompt symbol:

```
>
```

R will accept the command once it is syntactically complete; in particular any parentheses must balance. Once the command is complete, R will execute it.

Several commands can be given on the same line, separated by ;. A command may be interrupted by pressing the Esc key.

\(^2\) or cut-and-paste from a document such as this one
To illustrate this interaction, we draw a sample of random numbers from the uniform probability distribution; this is a simple example of R’s simulation repertoire.

Note: The code in these exercises was tested with Sweave [30, 31] on R version 3.3.2 (2016-10-31), sp package Version: 1.2-4, gstat package Version: 1.1-5, and lattice package Version: 0.20-35 running on Mac OS X 10.7.5. The text and graphical output you see here was automatically generated and incorporated into \LaTeX{} by running R source code through R and its packages. Then the R source is compiled into the PDF version you are now reading. Your output may be slightly different on different versions and on different platforms.

Task 2: Draw 12 random numbers uniformly distributed from -1 to 1, rounded to two decimal places, and sort them from smallest to largest.

In this tutorial we show the R code like this, with the prompt > and then then command:

> sort(round(runif(12, -1, 1), 2))

Note that the prompt > is not part of the command typed by the user; it is presented by the R console.

We show the output printed by R like this:

[1] -0.87 -0.65 -0.57 -0.49 -0.30 -0.25 -0.22 -0.19 0.13 0.17 0.28
[12] 0.81

The numbers in brackets, like [1], refer to the position in the output vector.

In the example above, the 12th element is 0.81.

This first example already illustrates several features of R:

1. It includes a large number of functions (here, \texttt{runif} to generate random numbers from the uniform distribution; \texttt{round} to round them to a specified precision; and \texttt{sort} to sort them);

2. These functions have arguments that specify the exact behaviour of the function. For example, \texttt{round} has two arguments: the first is the object to be rounded (here, the vector returned by the \texttt{runif} function) and the second is the number of decimal places (here, 2);

3. Many functions are vectorized: they can work on vectors (and usually matrices) as well as scalars. Here the \texttt{round} function is modifying the results of the \texttt{runif} function, which is a 12-element vector;

4. Values returned by a function can be immediately used as an argument to another function. Here the results of \texttt{runif} is the vector to be rounded by the \texttt{round} function; and these are then used by the \texttt{sort} function. To understand a complex expression, read it from the inside out.

5. R has a rich set of functions for simulation of random processes.
Q1: Your results will be different from the ones printed in this note; why?

To see how this works, we can do the same operation step-by-step.

1. Draw the random sample, and save it in a local variable in the workspace using the \(<-\) (assignment) operator; we also list it on the console with the `print` function:

   \[
   \begin{align*}
   \texttt{sample} & \leftarrow \text{runif(12, -1, 1)} \\
   \texttt{print(sample)}
   \end{align*}
   \]

   \[
   \begin{align*}
   [1] & 0.93315072 \ 0.13637810 \ -0.15049418 \ -0.56917349 \ 0.99437162 \\
   [6] & -0.00076908 \ -0.60171802 \ -0.81751347 \ -0.07128526 \ 0.10657717 \\
   [11] & 0.03592193 \ 0.42981844
   \end{align*}
   \]

2. Round it to two decimal places, storing it in the same variable (i.e. replacing the original sample):

   \[
   \begin{align*}
   \texttt{sample} & \leftarrow \text{round(sample, 2)} \\
   \texttt{sample}
   \end{align*}
   \]

   \[
   \begin{align*}
   [1] & 0.93 \ 0.14 \ -0.15 \ -0.57 \ 0.99 \ 0.00 \ -0.60 \ -0.82 \ -0.07 \ 0.11 \ 0.04 \\
   [12] & 0.43
   \end{align*}
   \]

3. Sort it and print the results:

   \[
   \begin{align*}
   \texttt{(sample} & \leftarrow \text{sort(sample)}) \\
   \texttt{(sample)}
   \end{align*}
   \]

   \[
   \begin{align*}
   [1] & -0.82 \ -0.60 \ -0.57 \ -0.15 \ -0.07 \ 0.00 \ 0.04 \ 0.11 \ 0.14 \ 0.43 \ 0.93 \\
   [12] & 0.99
   \end{align*}
   \]

This example also shows three ways of printing R output on the console:

- By using the `print` function with the object name as argument;
- By simply typing the object name; this calls the `print` function;
- By enclosing any expression in parenthesis ( . . . ); this forces another evaluation, which prints its results.

R has an immense repertoire of statistical methods; let’s see two of the most basic.

Task 3: Compute the theoretical and empirical mean and variance of a sample of 20 observations from a uniformly-distributed random variable in the range (0...10), and compare them.

The theoretical mean and variance of a uniformly-distributed random variable are [6, §3.3]:

\[
\begin{align*}
\mu & = (b + a)/2 \\
\sigma^2 & = (b - a)^2/12
\end{align*}
\]
where $a$ and $b$ are the lower and upper endpoints, respectively, of the uniform interval.

First the theoretical values for the mean and variance. Although we could compute these by hand, it’s instructive to see how R can be used as an interactive calculator with the usual operators such as $+$, $-$, $\times$, $/$, and $^\wedge$ (for exponentiation):

```r
> (10 + 0)/2
[1] 5
> (10 - 0)^2/12
[1] 8.3333
```

Now draw a 20-element sample and compute the sample mean and variance, using the `mean` and `var` functions:

```r
> sample <- runif(20, min = 0, max = 10)
> mean(sample)
[1] 4.8028
> var(sample)
[1] 10.744
```

---

**Q2:** How close did your sample come to the theoretical value?  
*Jump to A2* •

We are done with the local variable `sample`, so we remove it from the workspace with the `rm` (“remove”) function; we can check the contents of the workspace with the `ls` (“list”) function:

```r
> ls()
[1] "sample"
> rm(sample)
> ls()
character(0)
```

**On-line help**  If you know a function or function’s name, you can get `help` on it with the `help` function:

```r
> help(round)
```

This can also be written more simply as `?round`.

---

**Q3:** Use the `help` function to find out the three arguments to the `runif` function. What are these? Are they all required? Does the order matter?  
*Jump to A3* •
Arguments to functions  We can experiment a bit to see the effect of changing the arguments:

```r
> runif(1)
[1] 0.71507
> sort(runif(12))
[1] 0.036742 0.095620 0.317155 0.439924 0.550483 0.578844 0.631454
[8] 0.724262 0.821409 0.880349 0.959530 0.960348
> sort(runif(12, 0, 5))
[1] 0.59751 0.69794 1.09777 1.36984 1.63890 2.33485 2.68574 3.21069
> sort(runif(12, min = 0, max = 5))
[1] 0.11934 0.33598 0.43909 1.33396 1.61448 2.95719 3.19002 3.42839
> sort(runif(max = 5, n = 12, min = 0))
[1] 0.37364 1.07499 1.35853 1.37770 1.79783 3.15530 3.21438 3.22140
```

Searching for a function  If you don’t know a function name, but you know what you want to accomplish, you can search for an appropriate function with the `help.search` function:

```r
> help.search("principal component")
```

This will show packages and functions relevant to the topic:

- `stats::biplot.princomp`  Biplot for Principal Components
- `stats::prcomp`  Principal Components Analysis
- `stats::princomp`  Principal Components Analysis
- `stats::summary.princomp`  Summary method for Principal Components Analysis

Then you can ask for more information on one of these, e.g.:

```r
> help(prcomp)
```

```
prcomp package:stats R Documentation

Principal Components Analysis

Description:

Performs a principal components analysis on the given data matrix and returns the results as an object of class 'prcomp'.

Usage: ...
```
2.1 Leaving R

At this point you should leave R and re-start it, to see how that’s done.

Before leaving R, you may want to save your console log as a text file to document what you did, and the results, or for later re-use. You can edit this file in any plain-text editor, or include in a report.

To leave R, use the `q` (“quit”) function; if you are running R with a GUI, you can use a menu command, or a “close” icon as in any GUI program.

```R
> q()
```

You will be asked if you want to save your workspace in the current directory; generally you will want to do this³. The next time you start R in the same directory, the saved workspace will be automatically loaded.

In this case we haven’t created anything useful for data analysis, so you should quit without saving the workspace.

2.2 Answers

---

**A1 :** Random number generation gives a different result each time.⁴.  
Return to Q1 •

---

**A2 :** This depends on your sample; see the results in the text for an example.  
Return to Q2 •

---

**A3 :** There are three possible arguments: the number of samples `n`, the minimum value `min` and the maximum `max`. The last two are not required and default to 0 and 1, respectively. If arguments are named directly, they can be put in any order. If not, they have to follow the default order.  
Return to Q3 •

3 Loading and examining a data set

The remainder of this tutorial uses the Mercer & Hall wheat yield data set, which is described in Appendix A. Please read this now.

There are many ways to get data into R [44, §6]; one of the simplest is to create a comma-separated values ("CSV") file in a text editor⁵. For this example we have prepared file `mhw.csv` which is supplied with this tutorial.

**Task 4 :** From the operating system, open the text file `mhw.csv` with a plain-text editor such as WordPad and examine its structure.

*Do not* examine it in Excel; this automatically splits the file into spreadsheet columns, obscuring its structure as a text file.

---

³ By default this file is named `.RData`

⁴ To start a simulation at the same point (e.g. for testing) use the `set.seed` function

⁵ A CSV file can also be prepared as a spreadsheet and exported to CSV format.
The first four lines of the file should look like this:

```
r", "c", "grain", "straw"
1,1,3.63,6.37
2,1,4.07,6.24
3,1,4.51,7.05
```

**Q4**: What does the first line represent? What do the other lines represent, and what is their structure? Jump to A4 •

### 3.1 Reading a CSV file into an R object

Now we read the dataset into R.

**Task 5**: Start R.

**Task 6**: If necessary, make sure R is pointed to the same **working directory** where you have stored `mhw.csv`. You can use the `getwd` function to check this, and `setwd` to change it, or a menu command in your favourite IDE.

```r
> getwd()
```

Once the directory is changed, the contents of the file can be displayed with the `file.show` function:

```r
> file.show("mhw.csv")
```

```
r", "c", "grain", "straw"
1,1,3.63,6.37
2,1,4.07,6.24
3,1,4.51,7.05
4,1,3.9,6.91
...
```

A CSV file can be read into R with the `read.csv` function and **assigned** to an object in the **workspace** using the `<- operator** (which can also be written as `=`):

```r
> mhw <- read.csv("mhw.csv")
```

**Q5**: Why is nothing printed after this command? Jump to A5 •

### 3.2 Examining a dataset

The first thing to do with any dataset is to examine its **structure** with the `str` function.

```r
> str(mhw)
```
'data.frame': 500 obs. of 4 variables:
$ r : int 1 2 3 4 5 6 7 8 9 10 ...
$ c : int 1 1 1 1 1 1 1 1 1 1 ...
$ grain: num 3.63 4.07 4.51 3.9 3.63 3.16 3.18 3.42 3.97 3.4 ...
$ straw: num 6.37 6.24 7.05 6.91 5.93 5.59 5.32 5.52 6.03 5.66 ...

Q6: How many observations (cases) are there in this frame? How many fields (variables)? What are the field names?

We can extract the names for each field (matrix column) with the names function; this is equivalent to colnames:

> names(mhw)
[1] "r" "c" "grain" "straw"

> colnames(mhw)
[1] "r" "c" "grain" "straw"

Every object in R belongs to a class, which R uses to decide how to carry out commands.

Q7: What is the class of this object?

We can examine the class with the class function:

> class(mhw)
[1] "data.frame"

A data frame is used to hold most data sets. The matrix rows are the observations or cases; the matrix columns are the named fields or variables. Both matrix rows and columns have names.

Fields in the data frame are commonly referred to by their matrix column name, using the syntax frame$variable, which can be read as “extract the field named variable from the data frame named frame.”

Task 7: Summarize the grain and straw yields.

> summary(mhw$grain)

   Min. 1st Qu.  Median   Mean 3rd Qu.   Max.  
  2.73   3.64   3.94    3.95  4.27    5.16 

> summary(mhw$straw)

   Min. 1st Qu.  Median   Mean 3rd Qu.   Max.  
  4.10   5.88   6.36    6.51  7.17    8.85 

9
A data frame is also a **matrix**; we can see this by examining its **dimensions** with the `dim` function and extracting elements.

The two dimensions are the numbers of matrix rows and columns:

```r
> dim(mhw)
[1] 500 4
```

**Q8**: Which matrix dimension corresponds to the observations and which to the fields? **Jump to A8 •**

Matrix rows, columns, and individual cells in the matrix can be extracted with the `[]` operator; this is just like standard matrix notation in mathematics:

```r
> mhw[1, ]
 r c grain straw
 1 1 1 3.63 6.37

> length(mhw[, 3])
[1] 500

> summary(mhw[, 3])

    Min. 1st Qu.  Median    Mean 3rd Qu.   Max.   
    2.73  3.64  3.94  3.95  4.27  5.16

> mhw[1, 3]
[1] 3.63
```

Matrix rows and columns can also be accessed by their **names**; here is the grain yield of the first plot:

```r
> mhw[1, "grain"]
[1] 3.63
```

**Q9**: What is the grain yield of plot 64? Where is this located in the (experimental) field? **Jump to A9 •**

```r
> mhw[64, "grain"]
[1] 4.04

> mhw[64, c("r", "c")]
 r c
 64 4 4
```
Note the use of the \texttt{c} ("catenate", Latin for ‘build a chain’) function to build a \texttt{list} of two names.

Several adjacent rows or columns can be specified with the : “sequence” operator. For example, to show the row and column in the wheat field for the first three records:

\[
\texttt{mhw}[1:3, 1:2]
\]

\[
\begin{array}{ccc}
  r & c \\
  1 & 1 & 1 \\
  2 & 2 & 1 \\
  3 & 3 & 1 \\
\end{array}
\]

Rows or columns can be omitted with the – “minus” operator; this is shorthand for “leave these out, show the rest”. For example to summarize the grain yields for all except the first field column:\footnote{recall, the dataset is presented in field column-major order, and there are 20 field rows per field column}

\[
\texttt{summary(mhw[-(1:20), "grain"])}
\]

\begin{tabular}{lcccc}
  Min. & 1st Qu. & Median & Mean & 3rd Qu. & Max. \\
  2.73 & 3.64 & 3.94 & 3.95 & 4.27 & 5.16 \\
\end{tabular}

\[
\texttt{summary(mhw$grain[-(1:20)]})
\]

\begin{tabular}{lcccc}
  Min. & 1st Qu. & Median & Mean & 3rd Qu. & Max. \\
  2.73 & 3.64 & 3.94 & 3.95 & 4.27 & 5.16 \\
\end{tabular}

An entire field (variable) can be accessed either by matrix column number or name (considering the object to be a matrix) or variable name (considering the object to be a data frame); the output can be limited to the first and last lines only by using the \texttt{head} and \texttt{tail} functions. By default they show the six first or last values; this can be overridden with the optional \texttt{n} argument.

\[
\texttt{head(mhw[, 3])}
\]

\[
\]

\[
\texttt{tail(mhw[, "grain"], n = 10)}
\]

\[
\]

\[
\texttt{head(mhw$grain)}
\]

\[
\]

The \texttt{order} function is somewhat like the \texttt{sort} function shown above, but rather than return the actual values, it returns their position in the array. This position can then be used to extract other information from the data frame.

\textbf{Task 8} : Display the information for the plots with the five \textit{lowest} straw yields.
To restrict the results to only five, we again use the head function.

```r
data = mhw[]
head(sort(data$straw), n = 5)
#[1] 4.10 4.28 4.53 4.56 4.57
```

```r
data = mhw[]
head(order(data$straw), n = 5)
#[1] 470 467 441 447 427
```

```r
data[order(data$straw), , n = 5]
```

<table>
<thead>
<tr>
<th>r</th>
<th>c</th>
<th>grain</th>
<th>straw</th>
</tr>
</thead>
<tbody>
<tr>
<td>470</td>
<td>10</td>
<td>24</td>
<td>2.84</td>
</tr>
<tr>
<td>467</td>
<td>7</td>
<td>24</td>
<td>2.78</td>
</tr>
<tr>
<td>441</td>
<td>1</td>
<td>23</td>
<td>2.97</td>
</tr>
<tr>
<td>447</td>
<td>7</td>
<td>23</td>
<td>3.44</td>
</tr>
<tr>
<td>427</td>
<td>7</td>
<td>23</td>
<td>3.05</td>
</tr>
</tbody>
</table>

**Q10**: What are the values shown in the first command, using `sort`? In the second, using `order`? Why must we use the results of `order` to extract the records in the data frame? Jump to A10

**Task 9**: Display the information for the plots with the highest straw yields.

One way is to use the `rev` command to reverse the results of the `sort` or `order` function:

```r
> head(rev(sort(data$straw)), n = 5)
#[1] 8.85 8.85 8.78 8.75 8.74
```

Another way is to use the optional `decreasing` argument to `sort` or `order`; by default this has the value `FALSE` (so the sort is ascending); by setting it to `TRUE` the sort will be descending:

```r
> head(sort(data$straw, decreasing = T), n = 5)
#[1] 8.85 8.85 8.78 8.75 8.74
```

And a final way is to display the end of the ascending order vector, instead of the beginning, with the `tail` function; however, this shows the last records but still in ascending order:

```r
> tail(sort(data$straw), n = 5)
#[1] 8.74 8.75 8.78 8.85 8.85
```

Records can also be selected with logical criteria, for example with numeric comparison operators.
Task 10: Identify the plots with the highest and lowest grain yields and show their location in the field and both yields.

There are two ways to do this. First, apply the `max` and `min` functions to the grain yield field, and use their values (i.e., the highest and lowest yields) as a row selector, along with the `=="numerical equality"` comparison operator.

We save the returned value (i.e., the row number where the maximum or minimum is found), and then use this as the row subscript selector:

```r
> (ix <- which(mhw$grain == max(mhw$grain)))
[1] 79
> mhw[ix, ,]
   r  c grain straw
  79 19 4 5.16 8.78
> (ix <- which(mhw$grain == min(mhw$grain)))
[1] 338
> mhw[ix, ,]
   r  c grain straw
 338 18 17 2.73 4.77
```

The easier way, in the case of the minimum or maximum, is to use the `which.max` (index of the maximum value in a vector) and `which.min` (index of the minimum value in a vector) function

```r
> (ix <- which.max(mhw$grain))
[1] 79
> mhw[ix, ,]
   r  c grain straw
  79 19 4 5.16 8.78
> (ix <- which.min(mhw$grain))
[1] 338
> mhw[ix, ,]
   r  c grain straw
 338 18 17 2.73 4.77
```

Q11: Why is there nothing between the comma ‘,’ and right bracket ‘]’ in the expressions `mhw[ix, ,]` above?  

The advantage of the first method is that `==` or other numeric comparison operators can be used to select; operators include `!=` (not equal), `<`, `>`, `<=` (≤), and `>=` (≥). For example:
Task 11: Display the records for the plots with straw yield > 8.8 lb. per plot.

```r
> mhw[which(mhw$straw > 8.8), ]
```

<table>
<thead>
<tr>
<th>r</th>
<th>c</th>
<th>grain</th>
<th>straw</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1</td>
<td>3.46</td>
<td>8.85</td>
</tr>
<tr>
<td>98</td>
<td>18</td>
<td>4.84</td>
<td>8.85</td>
</tr>
</tbody>
</table>

Challenge: Extract all the grain yields from the most easterly (highest-numbered) column of field plots, along with the straw yields and field plot row number. Sort them from highest to lowest yields, also displaying the row numbers and straw yields. Does there seem to be any trend by field plot row? How closely are the decreasing grain yields matched by straw yields?

3.3 Saving a dataset in R format

Once a dataset has been read into R and possibly modified (for example, by assigning field names, changing the class of some fields, or computing new fields) it can be saved in R’s internal format, using the `save` function. The dataset can then be read into the workspace in a future session with the `load` function.

Task 12: Save the `mhw` object in R format.

```r
> save(mhw, file = "mhw.RData")
```

3.4 Answers

A4: The first line is a header with the variable names, in this case `r`, `c`, `grain` and `straw`. The following lines each represent one plot; there are four variables recorded for each plot, i.e. its row and column number in the field, and its grain and straw yield.  

A5: Commands that store their results in an object (using the `=` or `<-` operators) do their work silently; if you want to see the results enclose the command in parentheses ( ... ) or just type the object name at the command prompt.

A6: There are 500 observations (cases), and for each 4 variables: `r`, `c`, `grain` and `straw`.

A7: It is in class `data.frame`. 
A8: Matrix rows are observations, matrix columns are fields. Return to Q8

A9: Grain yield 4.04; this is located at field row 4, field column 4 Return to Q9

A10: The sort function shows the actual values of straw yield; order shows in which records in the data frame these are found. The record numbers are the key into the data frame. Return to Q10

A11: So that all fields (matrix columns) are selected. Return to Q11

4 Exploratory graphics

Before beginning a data analysis, it is helpful to visualise the dataset. This is generally the first phase of exploratory data analysis (EDA) [48].

R is an excellent environment for visualisation; it can produce simple plots but also plots of great sophistication, information and beauty. We look first at single variables and then at the relation between two variables.

4.1 Univariate exploratory graphics

Task 13: Visualise the frequency distribution of grain yield with a stem plot.

A stem-and-leaf plot, displayed by the stem function, shows the numerical values themselves, to some precision:

```r
> stem(mhw$grain)
The decimal point is 1 digit(s) to the left of the |
27 | 38
28 | 45
29 | 279
30 | 1445555578999
31 | 4446678999
32 | 2345589999
33 | 00245666677789999
34 | 00112233334444556666677778889999
35 | 01112334445555666667778999999
36 | 00011111333334444455666667777888899999
37 | 0001111122222222333344444555556666677778999999
38 | 00112222233334444445555566666677777999999
39 | 011111111222222333344444455555666666777777777999
40 | 01112223333444455556666668777777788888999999999
41 | 000111111223334444555555557777777777777777777777777777999
42 | 000011111112223333444444466666777777788889999999999
43 | 0111223333566666667777777788889999999999
44 | 001111122223444445556666677777899999
```
Q12: According to the stem-and-leaf plot, what are the approximate values of the minimum and maximum grain yields? Jump to A12

Q13: What is the advantage of the stem plot over the histogram? Jump to A13

Task 14: Visualise the frequency distribution of grain yield with a frequency histogram.

A histogram, displayed by the `hist` function, shows the distribution:

```R
> hist(mhw$grain)
```

Saving graphic output

You can save the graphics window to any common graphics format.

- In RStudio, click on the “Export” button in the “Plots” tab.
- In the Windows GUI, bring the graphics window to the front (e.g., click on its title bar), select menu command File | Save as ... and then one of the formats.
Q14: What are the two axes of the default histogram?  

Q15: By examining the histogram, how many observations had grain yield below 3 lb. per plot?

4.1.1 Enhancing the histogram*

R graphics, including histograms, can be enhanced from their quite plain default appearance. Here we change the break points with the `breaks` argument, the colour of the bars with the `col` graphics argument, the colour of the border with the `border` graphics argument, and supply a title with the `main` graphics argument.

We then use the `rug` function to add a “rug” plot along the x-axis to show the actual observations. This is an example of a graphics function that adds to an existing plot; whereas `hist` creates a new plot. Which does which? Consult the help.

```r
> hist(mhw$grain, breaks = seq(2.6, 5.2, by = 0.1), col = "lightblue", + border = "red", main = "Mercer-Hall uniformity trial", + xlab = "Grain yield, lb. per plot")
> rug(mhw$grain)
```

Note the use of the `seq` (“sequence”) function to make a list of break points:

```r
> seq(2.6, 5.2, by = 0.1)
```

```
[1] 2.6 2.7 2.8 2.9 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0 4.1
[17] 4.2 4.3 4.4 4.5 4.6 4.7 4.8 4.9 5.0 5.1 5.2
```

In this example, the colours are from a list of known names. For more information on these names, and other ways to specify colours, see Appendix B.
4.1.2 Kernel density*

A kernel density, computed with `density` function, fits an empirical curve to a sample supposed to be drawn from a univariate probability distribution [50, §5.6]. It can be used to give a visual impression of the distribution or to smooth an empirical distribution.

In the context of EDA, the kernel density can suggest:

- whether the empirical distribution is unimodal or multimodal;
- in the case of a unimodal distribution, the theoretical probability density function from which it may have been drawn.

The kernel density is controlled by the `kernel` and `adjust` optional arguments to the `density` function; see `?density` for details. The default values of "gaussian" and 1 select a smoothing bandwidth based on the number of observations and a theoretical normal density.

A special case of the density is a histogram expressed as densities rather than frequencies; this is selected with the optional `freq` ("frequency") argument to the `hist` function set to `FALSE`. The total area under the histogram is then by definition 1.

The `lines` function can be used to add the empirical density computed by `density` to a density histogram plotted with `hist`. Another interesting view is the kernel density with a rug plot to show the actual values of the sample.

**Task 15**: Display a histogram of the grain yields as a density (proportion of the total), with the default kernel density superimposed, along with a double and half bandwidth kernel density.

```r
> hist(mhw$grain, breaks = seq(2.6, 5.2, by = 0.1), col = "lavender", + border = "darkblue", main = "Mercer-Hall uniformity trial", + freq = F, xlab = "Grain yield, lb. per plot")
> lines(density(mhw$grain), lwd = 1.5)
> lines(density(mhw$grain, adj = 2), lwd = 1.5, col = "brown")
> lines(density(mhw$grain, adj = 0.5), lwd = 1.5, col = "red")
> text(2.5, 0.95, "Default bandwidth", col = "darkblue", + pos = 4)
> text(2.5, 0.9, "Double bandwidth", col = "brown", pos = 4)
> text(2.5, 0.85, "Half bandwidth", col = "red", pos = 4)
```
Task 16: Repeat, but show just the kernel density with a rug plot (i.e. no histogram).

Here the first plot must be of the density, because `rug` only adds to an existing plot.

```r
> plot(density(mhw$grain) , xlab="Grain yield, lb. per plot", +     lwd=1.5, ylim=c(0,1), col="darkblue", +     main="Mercer-Hall uniformity trial")
> rug(mhw$grain)
> lines(density(mhw$grain, adj=2), lwd=1.5, col="brown")
> lines(density(mhw$grain, adj=.5), lwd=1.5, col="red")
> text(2.5,0.85,"Default bandwidth", col="darkblue", pos=4)
> text(2.5,0.80,"Double bandwidth", col="brown", pos=4)
> text(2.5,0.75,"Half bandwidth", col="red", pos=4)
```
Q16: Which bandwidths give rougher or smoother curves? What does the curve for the default bandwidth suggest about the underlying distribution?

4.1.3 Another histogram enhancement: colour-coding relative frequency*

Task 17: Display a histogram of the grain yield with break points every 0.2 lb., with the count in each histogram bin printed on the appropriate bar. Shade the bars according to their count, in a colour ramp with low counts whiter and high counts redder.

The solution to this task depends on the fact that the `hist` function not only plots a histogram graph, it can also return an object which can be assigned to an object in the workspace; we can then examine the object to find the counts, breakpoints etc. We first compute the histogram but don’t plot it (`plot=F` argument), then draw it with the `plot` command, specifying a colour ramp, which uses the computed counts, and a title. Then the `text` command adds text to the plot at (x, y) positions computed from the class mid-points and counts; the `pos=3` argument puts the text on top of the bar.

```r
> h <- hist(mhw$grain, breaks = seq(2.6, 5.2, by=.2), plot=F)
> str(h)
```

List of 6

- `$breaks` : num [1:14] 2.6 2.8 3 3.2 3.4 3.6 3.8 4 4.2 4.4 ...
- `$counts` : int [1:13] 2 5 22 30 56 80 79 73 70 48 ...
- `$density` : num [1:13] 0.02 0.05 0.22 0.3 0.56 ...
- `$mids` : num [1:13] 2.7 2.9 3.1 3.3 3.5 3.7 3.9 4.1 4.3 4.5 ...
- `$xname` : chr "mhw$grain"
$ \text{equidist: logi TRUE}$
- `attr(*, "class")= chr "histogram"`

```r
> plot(h, col = heat.colors(length(h$mids))[(length(h$count)-
+ rank(h$count)+1)],
+ ylim = c(0, max(h$count)+5),
+ main="Frequency histogram, Mercer & Hall grain yield",
+ sub="Counts shown above bar, actual values shown with rug plot",
+ xlab="Grain yield, lb. per plot")
> rug(mhw$grain)
> text(h$mids, h$count, h$count, pos=3)
> rm(h)
```

**4.2 Bivariate exploratory graphics**

When several variables have been collected, it is natural to compare them.

**Task 18**: Display a **scatterplot** of straw vs. grain yield.

We again use `plot`, but in this case there are two variables, so a scatterplot is produced. That is, `plot` is an example of a **generic** function: its behaviour changes according to the **class** of object it is asked to work on.

```r
> plot(mhw$grain, mhw$straw)
```
**Q17:** What is the relation between grain and straw yield?  

Jump to A17 •

This plot can be enhanced with much more information. For example:

- We add a grid at the axis ticks with the `grid` function;
- We specify the plotting character with the `pch` graphics argument,
- its colours with the `col` (outline) and `bg` (fill) graphics arguments,
- its size with the `cex”character expansion” graphics argument,
- the axis labels with the `xlab` and `ylab` graphics arguments;
- We add a title with the `title` function, and
- mark the centroid (centre of gravity) with two calls to `abline`, one specifying a vertical line (argument `v=`) and one horizontal (argument `vh=`) at the means of the two variables, computed with the `mean` function;
- The two lines are dashed, using the `lty”line type” graphics argument,
- and coloured red using `col`;
- The centroid is shown as large diamond, using the `points` function and the `cex` graphics argument;
- Finally, the actual mean yields are displayed with the `text` function, using the `pos` and `adj` graphic argument to position the text with respect to the plotting position.

```r
> plot(mhw$grain, mhw$straw, cex=0.8, pch=21, col="blue",
+     bg="red", xlab="Grain yield, lb./ plot-1",
+     ylab="Straw yield, lb./ per plot-1")
> grid()
```
> title(main="Mercer-Hall wheat uniformity trial")
> abline(v=mean(mhw$grain), lty=2, col="blue")
> abline(h=mean(mhw$straw), lty=2, col="blue")
> points(mean(mhw$grain), mean(mhw$straw), pch=23, col="black",
+   bg="brown", cex=2)
> text(mean(mhw$grain), min(mhw$straw),
+   paste("Mean:",round(mean(mhw$grain),2)), pos=4)
> text(min(mhw$grain), mean(mhw$straw),
+   paste("Mean:",round(mean(mhw$straw),2)), adj=c(0,-1))

The advantage of this **programmed enhancement** is that we can store the commands as a script and reproduce the graph by running the script.

Some R graphics allow **interaction**.

---

**Task 19**: Identify the plots which do not fit the general pattern. (In any analysis these can be the most interesting cases, requiring explanation.)

For this we use the **identify** function, specifying the same plot coordinates as the previous **plot** command (i.e. from the plot that is currently displayed):

```r
> plot(mhw$grain, mhw$straw)
> pts <- identify(mhw$grain, mhw$straw)
```

After **identify** is called, switch to the graphics window, **left-click** with the mouse on points to identify them, and **right-click** to exit. The plot should now show the row names of the selected points:
Q18: Which observations have grain yield that is much lower than expected (considering the straw) and which higher?  

Jump to A18 •

\[
\begin{align*}
\text{r} & \quad \text{c} & \quad \text{grain} & \quad \text{straw} \\
337 & \quad 17 & \quad 17 & \quad 3.05 & \quad 7.64 \\
15 & \quad 15 & \quad 1 & \quad 3.46 & \quad 8.85 \\
295 & \quad 15 & \quad 15 & \quad 3.73 & \quad 8.58 \\
311 & \quad 11 & \quad 16 & \quad 3.74 & \quad 8.63 \\
284 & \quad 4 & \quad 15 & \quad 3.75 & \quad 4.62 \\
35 & \quad 15 & \quad 2 & \quad 4.42 & \quad 5.20 \\
184 & \quad 4 & \quad 10 & \quad 4.59 & \quad 5.41 \\
292 & \quad 12 & \quad 15 & \quad 4.86 & \quad 6.39 \\
\end{align*}
\]

> rm(pts, tmp)

4.3 Answers

A12: 2.73 and 5.16 lb. per plot, respectively. Note the placement of the decimal point, as explained in the plot header. Here it is one digit to the left of the |, so the entry 27 | 38 is to be read as 2.73, 2.78.  

Return to Q12 •

A13: The stem plot shows the actual values (to some number of significant digits). This allows us to see if there is any pattern to the digits.  

Return to Q13 •

A14: The horizontal axis is the value of the variable being summarized (in this case, grain yield). It is divided into sections (“histogram bins”) whose limits are
shown by the vertical vars. The vertical axis is the count (frequency) of observations in each bin.

A15: The two left-most histogram bins represent the values below 3 lb. per plot (horizontal axis); these appear to have 2 and 5 observations, respectively, for a total of 7, although it’s difficult to estimate exactly. The stem plot, which shows the values to some precision, can show this exactly.

A16: The higher value of the adj argument to the density function gives a smoother curve. In this case with adj=2 the curve is indistinguishable from a univariate normal distribution. The default curve is quite similar but with a slight asymmetry (peak is a bit towards the smaller values) and shorter tails. But, considering the sample size, it still strongly suggests a normal distribution.

A17: They are positively associated: higher grain yields are generally associated with higher straw yields. The relation appears to be linear across the entire range of the two measured variables. But the relation is diffuse and there are some clear exceptions.

A18: Plots 15, 337, 311 and 295 have grain yields that are lower than the general pattern; plots 308, 292, 184 and 35 the opposite.

5 Descriptive statistics

After visualizing the dataset, the next step is to compute some numerical summaries, also known as descriptive statistics. We can summarize all the variables in the dataset at the same time or individually with the summary function:

```r
c> summary(mhw)
  r  c  grain  straw
  Min. :1.00 Min. :1 Min. :2.73 Min. :4.10
  1st Qu.:5.75 1st Qu.:7 1st Qu.:3.64 1st Qu.:5.88
  Median :10.50 Median :13 Median :3.94 Median :6.36
  Mean :10.50 Mean :13 Mean :3.95 Mean :6.51
  3rd Qu.:15.25 3rd Qu.:19 3rd Qu.:4.27 3rd Qu.:7.17
  Max. :20.00 Max. :25 Max. :5.16 Max. :8.85

c> summary(mhw$grain)
  Min. 1st Qu. Median  Mean 3rd Qu. Max.
  2.73  3.64  3.94  3.95  4.27  5.16
```

Q19: What are the summary statistics for grain yield?
5.1 Other descriptive statistics

The descriptive statistics in the summary all have their individual functions: \( \text{max} \), \( \text{median} \), \( \text{mean} \), \( \text{max} \), and \( \text{quantile} \). This latter has a second argument, \( \text{probs} \), with a single value or list (formed with the \( \text{c} \) function) of the probabilities for which the quantile is requested:

```r
> max(mhw$grain)
[1] 5.16
> min(mhw$grain)
[1] 2.73
> median(mhw$grain)
[1] 3.94
> mean(mhw$grain)
[1] 3.9486
> quantile(mhw$grain, probs = c(0.25, 0.75))
   25%  75%
2.7375 4.2700
```

Other statistics that are often reported are the variance, standard deviation (square root of the variance) and inter-quartile range (IQR).

**Task 20**: Compute these for grain yield.

The \( \text{var} \), \( \text{sd} \) and \( \text{IQR} \) functions compute these.

```r
> var(mhw$grain)
[1] 0.21002
> sd(mhw$grain)
[1] 0.45828
> IQR(mhw$grain)
[1] 0.6325
```

Other measures applied to distributions are the skewness (deviation from symmetry; symmetric distributions have no skewness) and kurtosis (concentration around central value; a normal distribution has kurtosis of 3). Functions for these are not part of base R but are provided as the \textit{skewness} and \textit{kurtosis} functions of the curiously-named \textit{e1071} package from the Department of Statistics, Technical University of Vienna\(^7\).

\(^7\)This may have been the Department’s administrative code.
Task 21: Load the e1071 package and compute the skewness and kurtosis. If e1071 is not already installed on your system, you have to install it first.

Optional packages are best loaded with the require function; this ensures that the package is not already loaded before loading it (to avoid duplication).

Note: Since this is the only use we make of this package, we can unload it with the detach function. This is generally not necessary.

```r
> require(e1071)
> skewness(mhw$grain)
[1] 0.035363
> kurtosis(mhw$grain)
[1] -0.27461
> detach(package:e1071)
```

The kurtosis function as implemented in e1071 computes the so-called “excess” kurtosis, i.e., the difference from the normal distribution’s value (3). So a normally-distributed variable would have no excess kurtosis.

Q20: How do the skew and kurtosis of this distribution compare to the expected values for a normally-distributed variable, i.e., 0 (skew) and 0 (excess kurtosis)? Jump to A20

5.2 A closer look at the distribution

Q21: What is the range in grain yields? What proportion of the median yield is this? Does this seem high or low, considering that all plots were treated the same? Jump to A21

To answer this we can use the diff (“difference”) and median functions:

```r
> diff(range(mhw$grain))
[1] 2.43
> diff(range(mhw$grain))/median(mhw$grain)
[1] 0.61675
```

Q22: Which is the lowest-yielding plot? Does it also have low straw yield? Jump to A22
To answer this, use the `which.min` function to identify the record number with the lowest yield:

```r
> ix <- which.min(mhw$grain)
> mhw[ix, "straw"]
[1] 4.77
```

We can select cases based on logical criteria, for example, to find the lowest-yielding plots.

**Task 22**: Find all observations with grain yield less than 3 lb. per plot, and also those with grain yield in the lowest (first) percentile.

We can use either the `subset` function or direct matrix selection. The `quantile` function returns a list with quantiles; here we illustrate the default, the case where we use the `seq` function to ask for the ten deciles, and finally just the 1% quantile:

```r
> row.names(subset(mhw, mhw$grain < 3))
[1] "149" "336" "338" "339" "441" "467" "470"
> quantile(mhw$grain)
0% 25% 50% 75% 100%
2.7300 3.6375 3.9400 4.2700 5.1600
> quantile(mhw$grain, seq(0, 1, 0.1))
0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%
> mhw[mhw$grain < quantile(mhw$grain, 0.01),]
   r  c  grain straw
336 16 17 2.92 4.95
338 18 17 2.73 4.77
339 19 17 2.85 4.96
467 7 24 2.78 4.28
470 10 24 2.84 4.10
```

**Q23**: Which plots have grain yield less than 3 lb.? Which are the lowest-yielding 1%? Are those close to each other?  

**5.3 Answers**

**A19**: Minimum 2.73, maximum 5.16, arithmetic mean 3.95, first quartile 3.64, third quartile 4.27, median 3.94.  

*Jump to A23*

*Return to Q19*
A20: Both skew and excess kurtosis are quite close to the expected values (0). This strengthens the evidence of a normal distribution. Return to Q20

A21: The range in grain yields is 2.43, which is about 62% of the median. This seems quite high considering the “equal” treatment. Return to Q21

A22: The lowest-yielding plot is 338, with a grain yield of 4.77 lb. Return to Q22

A23: Plots with yield less than 3 lb. are 149, 336, 338, 339, 441, 467, and 470. The lowest percent are plots 336, 338, 339, 467, and 470. The first three are all in field column 17 and almost adjacent field rows (16, 18, 19); this seems definitely to be a ‘low-yield “hot spot” in the experimental field. The last two are both in field column 24 but a few rows apart (7 and 10). Return to Q23

6 Editing a data frame

If you need to fix up a few data entry errors, the data frame can be edited interactively with the fix function:

> fix(mhw)

In this case there is nothing to change, so just close the editor.

New variables are calculated in the local variable space. For example, the grain-to-straw ratio is an important indicator of how well the wheat plant has formed grain, relative to its size.

Task 23: Compute the grain/straw ratio and summarize it. •

Arithmetic operations are performed on entire vectors; therefore these are called vectorized operations. Here the division (symbolized of course by /) divides each grain yield by the straw yield.

> gsr <- mhw$grain/mhw$straw
> summary(gsr)

     Min. 1st Qu.  Median    Mean  3rd Qu.   Max.  
    0.391  0.574  0.604   0.611  0.642  0.850

Q24: What is the range in grain/straw ratio? Is it relatively larger or smaller than the range in grain? Jump to A24 •

> range(gsr)

[1] 0.39096 0.85000

> diff(range(gsr))/median(gsr)

[1] 0.75944
> diff(range(mhw$grain))/median(mhw$grain)

[1] 0.61675

For further analysis we would like to include this in the data frame itself, as an additional variable.

---

**Task 24**: Add grain-straw ratio to the mhw data frame and remove it from the local workspace.

For this we use the `cbind` (“column bind”) function to add a new matrix column (data frame field):

```r
> mhw <- cbind(mhw, gsr)
> str(mhw)

'data.frame': 500 obs. of 5 variables:
$ r : int 1 2 3 4 5 6 7 8 9 10 ...
$ c : int 1 1 1 1 1 1 1 1 1 1 ...
$ grain: num 3.63 4.07 4.51 3.9 3.63 3.16 3.18 3.42 3.97 3.4 ...
$ straw: num 6.37 6.24 7.05 6.91 5.93 5.59 5.32 5.52 6.03 5.66 ...
$ gsr : num 0.57 0.652 0.64 0.564 0.612 ...
```

By default, the new matrix column is automatically given the name from the local variable.

Now we remove the local variable `gsr` so that we do not confuse it with the `gsr` field of the `mhw` data frame:

```r
> ls()
[1] "gsr" "ix" "mhw"
> rm(gsr)
> ls()
[1] "ix" "mhw"
```

---

**Task 25**: Save the updated `mhw` object in R format.

We use a different file name to distinguish this from the original file, without the added column.

```r
> save(mhw, file = "mhw2.RData")
```

6.1 Answers

**A24**: The range is from 0.39 to 0.85, i.e. the ratio of grain to straw doubles. This is about 76% of the median ratio, which is considerably higher than the comparable figure for grain yield (about 62%).

*Return to Q24*
After descriptive statistics and visualisation comes the attempt to build statistical models of the underlying processes. These are empirical mathematical relations that describe one variable in terms of some hypothetical underlying distribution of which it is a realisation, or describe several variables either as equals (“correlation”) or where one is described in terms of others (“regression”). We explore these, from simple to complex.

The simplest kind of model is about the distribution of a single variable, i.e., univariate modelling.

We suppose that the observed sample distribution is from an underlying probability distribution. This raises two questions: (1) what is the form of that distribution, and (2) what are its parameters?

To decide what theoretical distribution might fit, we first visualise the empirical distribution. This continues the ideas from §4.1.2.

**Task 26**: Visualise an empirical continuous frequency distribution on the rug plot.

We again use the `density` function, with default arguments:

```r
> plot(density(mhw$grain), col="darkblue",
+     main="Grain yield, lb. per plot", lwd=1.5)
> rug(mhw$grain, col="darkgreen")
> grid()
```

We can also view the distribution as a cumulative rather than density distribution.
Task 27: Visualise the empirical cumulative distribution of grain yield.

We use the `ecdf` ("empirical cumulative distribution function") function to compute the distribution, then plot it with the `plot` function. Vertical lines are added to the plot with the `abline` ("add a straight line") function, at the median, extremes, and specified quantiles.

```r
> plot(ecdf(mhw$grain), pch=1,
+     xlab="Mercer & Hall, Grain yield, lb. per plot",
+     ylab="Cumulative proportion of plots",
+     main="Empirical CDF",
+     sub="Quantiles shown with vertical lines")
> q <- quantile(mhw$grain, c(.05, .1, .25, .75, .9, .95))
> abline(v=q, lty=2)
> abline(v=median(mhw$grain), col="blue")
> abline(v=max(mhw$grain), col="green")
> abline(v=min(mhw$grain), col="green")
> text(q, 0.5, names(q))
> rm(q)
```

Q25: From the histogram, stem-and-leaf plot, empirical cumulative distribution, and theory, what probability distribution is indicated for the grain yield? 

We can also visualise the distribution against the theoretical normal distribution computed with the sample mean and variance. There are (at least) two useful ways to visualise this.

First, compare the actual and theoretical normal distribution is with the `qqnorm` function to plot these against each other and then superimpose the theoretical line with the `qqline` function:

```r
> qqnorm(mhw$grain, main = "Normal probability plot, grain yields (lb. plot-1)")
```
Q26: Does this change your opinion of normality? Jump to A26

The second way to visually compare an empirical and theoretical distribution is to display the empirical density plot, superimposing the normal distribution that would be expected with the sample mean and standard deviation.

Task 28: Fit a normal probability distribution to the empirical distribution of grain yield.

Q27: What are the best estimates of the parameters of a normal distribution for the grain yield? Jump to A27

These are computed with the mean and sd functions:

> mean(mhw$grain)

[1] 3.9486

> sd(mhw$grain)

[1] 0.45828

With these in hand, we can plot the theoretical distribution against the empirical distribution:

Task 29: Graphically compare the theoretical and empirical distributions.
We can also see this on the empirical density. This version also uses the `curve` method to draw the theoretical curve.
There are several tests of normality; here we use the Shapiro-Wilk test, implemented by the `shapiro.test` function. This compares the empirical distribution (from the sample) with the theoretical distribution, and computes a statistic ("W") for which is known the probability that it could occur by change, assuming the sample is really from a normally-distributed population. The reported probability value is the chance that rejecting the null hypothesis \( H_0 \) that the sample is from a normal population is an incorrect decision (i.e. the probability of committing a Type I error).

```r
> shapiro.test(mhw$grain)
Shapiro-Wilk normality test

data:  mhw$grain
W = 0.997, p-value = 0.49
```

Q28: According to the Shapiro-Wilk test, what is the probability if we reject the null hypothesis that this empirical distribution is a realisation of a normal distribution with the sample mean and standard deviation as parameters, we would be wrong (i.e. commit a Type I error)? So, should we reject the null hypothesis of normality? Should we consider grain yield to be a normally-distributed variable?

Once we’ve established that the yields can be reasonably modelled by a normal distribution, we can compute confidence limits on the mean yield. Mercer and Hall used the 50% confidence level, which they called the probable error: there is equal chance for the true yield to be inside as outside this interval.

**Task 30**: Compute the probable error for grain yield in a 1/500 acre plot.
Although we’ve fitted a normal distribution, both the mean and standard deviation were estimated from the sample. So, we should use Student’s $t$-distribution (although with so many plots the difference with the normal $z$-distribution will be very small).

The probable error is computed from the limits of the interquartile range of the distribution; we get the quantiles of the $t$ distribution with the `qt` function, specifying the degrees of freedom:

```r
> (t.q13 <- qt(c(0.25, 0.75), length(mhw$grain) - 1))
[1] -0.67498 0.67498
```

This is for $\mu = 0, \sigma = 1$. We then scale this to the data:

```r
> (pe <- mean(mhw$grain) + t.q13 * sd(mhw$grain))
[1] 3.6393 4.2580
```

And we can express it as a relative error; conventionally this is expressed as error $\pm$ relative to the mean:

```r
> rel.error <- (diff(pe)/2)/mean(mhw$grain)
> round(100 * rel.error, 2)
[1] 7.83
```

**Q29**: If the true yield over the entire field is the observed mean yield, what yields can be expected in any one 1/500 acre plot, with 50% confidence that the mean is in this range?  

**Q30**: What is the probable error expressed as a percentage of mean yield?

To put this in practical terms, since this was Mercer and Hall’s main concern, we calculate the probable error in absolute kg ha$^{-1}$.

We saw above that the mean grain yield in this experiment was 3.95 lb. plot$^{-1}$. Scaling this up to a hectare basis (which is how yields are normally expressed):

```r
> (yield.ha <- mean(mhw$grain) * 500/(0.40469)/(2.2046226))
[1] 2212.9
```

The two constants in this formula are 0.40469 ha acre$^{-1}$ and 2.2046226 lb. kg$^{-1}$, and there are 500 plots per acre.

**Q31**: If we try to estimate the yield in kg ha$^{-1}$ from a single 1/500 acre plot in a variety trial, how much error (in absolute terms) can we expect,
with 50% probability? What are the practical implications of this? Jump to A31

> round(yield.ha * rel.error, 2)

[1] 173.35

Clean up:

> rm(t.q13, pe, rel.error, yield.ha)

7.1 Answers

A25: The Normal distribution. From theory: the addition of multiple independent sources of noise. From the plots: visual fit to simulated normal distributions. Return to Q25

A26: The distribution still looks normal, except at both tails: the highest yields are not as high, and the lowest not as low, as expected if the yields were normally distributed. Return to Q26

A27: Parameters $\mu = 3.95$, $\sigma = 0.458$. Return to Q27

A28: According to the Shapiro-Wilk test of normality, the probability that rejecting the null hypothesis of normality would be an incorrect decision (i.e. a Type I error) is 0.49; this is quite high, so we do not reject the null hypothesis. So, we consider the grain yield to be a realisation of a normal distribution. Return to Q28

A29: From 3.64 to 4.36 lb. plot$^{-1}$; the observed mean yield is 3.95 lb. plot$^{-1}$. Return to Q29

A30: $\pm 7.83\%$. Return to Q30

A31: From a true yield (estimated here from our mean) of 2213 kg wheat grain, the probable error is 173 kg, a substantial amount. There is a 50% chance of observing this much deviation if we estimate the per-hectare yield from any single 1/500 acre plot. Clearly, this is where the idea of replicated plots originated. Return to Q31

8 Bivariate modelling: two continuous variables

After modelling the distribution of a single variable, we now model the joint distribution of two variables, i.e., bivariate modelling.

In §4.2 we displayed a scatterplot of straw vs. grain yield, repeated here with some additional graphical elements:
The ~ formula operator is used to indicate statistical dependence, here of straw on grain; we will explore this further just below. The centroid is shown by intersecting horizontal and vertical lines, added with the abline function.

Q32: From the scatterplot of straw vs. grain yield, what functional relation is suggested?  

For a bivariate linear relation, as hypothesised here, we can view this four ways:

1. A bivariate linear correlation between the two variables (straw and grain yields) (§8.1);
2. A univariate linear regression of straw (dependent) on grain (independent) yield (§8.2);
3. A univariate linear regression of grain (dependent) on straw (independent) yield (§8.3).
4. A linear structural relation between the two yields (§8.3).

These will each be explored below.

Bivariate correlation, bivariate structural relations and univariate regression
all compare two variables that refer to the same observations, that is, they are paired. This is the natural order in a data frame: each row represents one observation on which several variables were measured; in the present case, the row and column numbers of the plots, and the grain and straw yields.

Correlation and various kinds of regression are often misused. There are several good journal articles that explain the situation, with examples from earth science applications [34, 51]. A particularly understandable introduction to the proper use of regression is by Webster [52].

Note that in this case there is no evidence to suggest a non-linear or piecewise relation; but in many cases these are possibilities that must be compared with a linear model.

8.1 Correlation

Correlation measures the strength of the association between two variables measured on the same object, from −1 (perfect negative correlation), through 0 (no correlation), to +1 (perfect positive correlation). The two variables have logically equal status, so there is no concept of causation; in addition, there is no functional relation, so there is no way to predict.

There are several numerical measures of correlation; we will see two:

1. Parametric: Pearson’s product moment correlation coefficient (PMCC) \( r \) (§8.1.1);
2. Non-parametric: Spearman’s \( \rho \) (§11.3).

8.1.1 Parametric correlation

The PMCC should only be used if the two variables are distributed approximately bivariate normally. This is two normal distributions, but with some degree of correlation. So we first check whether the relation between grain and straw yield has this distribution.

Task 31: Visualise a bivariate normal distribution with the parameters of the grain and straw yields; visually compare with the actual bivariate distribution.

R has a \texttt{rnorm} function to simulate a random sample of a given size from the normal distribution, by analogy to the \texttt{runif} function presented above. However, to simulate a correlated sample of several variables, we turn to the \texttt{mvnorm} function in the \texttt{MASS} (“Modern Applied Statistics with S”) package which corresponds to the very useful advanced text of Venables and Ripley [50]. This function uses a vector of the variable means, along with the variance-covariance matrix of two or more variables.

The variable means are computed with the vectorized \texttt{colMeans} function, which finds the by-column mean all or some of the columns in a data frame:
The variance-covariance matrix is computed with the vectorized `var` function:

```r
> var(mhw[, c("grain", "straw")])
```

```
grain straw
grain 0.21002 0.30043
straw 0.30043 0.80696
```

Q33 : What do the diagonals and off-diagonals of this matrix represent? What are their units of measure? Jump to A33 •

From these the `mvnrmv` can draw a simulated random sample. We first load the optional `MASS` package with the `require` function; the `mvnrmv` function is then available.

```r
> require(MASS)
> sim.sample <- mvrnorm(length(mhw$grain), mu = colMeans(mhw[, + c("grain", "straw")]), Sigma = var(mhw[, c("grain", + "straw")]))
> head(sim.sample)
```

```
grain straw
[1,] 3.3667 5.0604
[2,] 3.9434 5.8178
[3,] 3.6377 6.1686
[4,] 3.8145 6.7367
[5,] 3.8117 6.6381
[6,] 3.3010 4.9470
```

```r
> summary(sim.sample)
```

```
grain straw
Min. :2.56 Min. :4.12
1st Qu.:3.67 1st Qu.:5.94
Median :3.95 Median :6.50
Mean :3.93 Mean :6.51
3rd Qu.:4.24 3rd Qu.:7.12
Max. :5.01 Max. :8.72
```

```r
> summary(mhw[, c("grain", "straw")])
```

```
grain straw
Min. :2.73 Min. :4.10
1st Qu.:3.64 1st Qu.:5.88
Median :3.94 Median :6.36
Mean :3.95 Mean :6.51
3rd Qu.:4.27 3rd Qu.:7.17
Max. :5.16 Max. :8.85
```

40
We can also visualise these distributions with histograms. To ensure that the visualisation is comparable, we compute the overall minimum and maximum of both variables and use these to explicitly set the axis limits.

```r
> par(mfrow = c(2, 2))
> grain.lim = c(min(sim.sample[, "grain"], mhw$grain),
+ max(sim.sample[, "grain"], mhw$grain))
> straw.lim = c(min(sim.sample[, "straw"], mhw$straw),
+ max(sim.sample[, "straw"], mhw$straw))
> hist(mhw$grain, xlim = grain.lim, main = "Grain (actual)",
+ col = "lightyellow", breaks = seq(grain.lim[1], grain.lim[2],
+ length = 17))
> hist(sim.sample[, "grain"], xlim = grain.lim, main = "Grain (simulated)",
+ col = "cornsilk", breaks = seq(grain.lim[1], grain.lim[2],
+ length = 17))
> hist(mhw$straw, xlim = straw.lim, main = "Straw (actual)",
+ col = "lightblue", breaks = seq(straw.lim[1], straw.lim[2],
+ length = 17))
> hist(sim.sample[, "straw"], xlim = straw.lim, main = "Straw (simulated)",
+ col = "lavender", breaks = seq(straw.lim[1], straw.lim[2],
+ length = 17))
> par(mfrow = c(1, 1))
```
Q34: How well do the two univariate simulations match the actual data?  

So we can simulate a sample with the same mean and standard deviation as the grain and straw yields, and plot them against each other in a scatterplot. We display this side-by-side with the actual scatterplot.

```r
> par(mfrow = c(1, 2))
> plot(sim.sample, main = "Simulated straw vs. grain yields",
+      xlab = "Grain (lb. plot−1)", ylab = "Straw (lb. plot−1)",
+      xlim = grain.lim, ylim = straw.lim, pch = 20, col = "blue")
> abline(v = median(sim.sample[, 1]), lty = 2, col = "red")
> abline(h = median(sim.sample[, 2]), lty = 2, col = "red")
> plot(mhw$grain, mhw$straw, main = "Actual straw vs. grain yields",
+      xlab = "Grain (lb. plot−1)", ylab = "Straw (lb. plot−1)",
+      xlim = grain.lim, ylim = straw.lim, pch = 20, col = "black")
> abline(v = median(mhw$grain), lty = 2, col = "red")
> abline(h = median(mhw$straw), lty = 2, col = "red")
> par(mfrow = c(1, 1))
```

Q35: Do the two relations (simulated vs. actual) look similar? Do they support the hypothesis of a bivariate linear relation?  

We delete the temporary variables used in this visualisation:

```r
> rm(sim.sample, grain.lim, straw.lim)
```

Challenge: The single simulation is only one realisation of the hypothetical process. Repeat the simulation several times and compare (1) the simulations with each other; (2) the simulations with the actual data.

With this evidence of bivariate normality, we are justified in computing the parametric correlation.
Task 32: Compute the PMCC between grain and straw yield, and its 95% confidence limit.

The `cor` function computes the correlation; the `cor.test` function also computes the confidence interval:

```r
> cor(mhw$grain, mhw$straw)
[1] 0.72978

> cor.test(mhw$grain, mhw$straw)

Pearson's product-moment correlation
data: mhw$grain and mhw$straw
t = 23.8, df = 498, p-value < 2e-16
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
  0.68599 0.76830
sample estimates:
cor
  0.72978
```

Q36: What are the lowest possible, most probable, and highest possible values of the correlation coefficient \( r \), with 95% confidence? Do you consider this a very strong, strong, moderately strong, weak, or no correlation? Positive or negative?  

Jump to A36 •

8.2 Univariate linear regression

Univariate linear regression differs from bivariate correlation in that one of the two variables is considered mathematically dependent on the other.

An important distinction is made between predictors which are known without error, whether fixed by the experimenter or measured, and those that are not.

Fixed effects model Webster [52] calls the first type a “Gauss linear model” because only the predictand has Gaussian or “normal” error, whereas the predictor is known without error. The regression goes in one direction only, from the mathematical predictor to the random response, and is modelled by a linear model with error in the response:

\[
y_i = B X_i + \varepsilon_i
\]  

of which the simplest case is a line with intercept:

\[
y_i = \beta_0 + \beta_1 x_i + \varepsilon_i
\]  

8 referring to the developer of least-squares regression
In this model there is no error associated with the predictors \( x_i \), only with the predictand \( y_i \). The predictors are known without error, or at least the error is quite small in comparison to the error in the model. An example is a designed agricultural experiment where the quantity of fertiliser added (the predictor) is specified by the design and the crop yield is measured (the predictand); there is random error \( \varepsilon_i \) in this response.

Further, the errors \( \varepsilon_i \) are considered to be **identically and independently distributed** (IID):

- no relation between the magnitude of the error and that of the predictor (homoscedasticity);
- no serial correlation between errors (e.g., small errors systematically followed by another small errors) in the sequence of predictors.

These assumptions can be verified after the regression parameters are estimated, using feature-space **regression diagnostics** (§8.2.4) and spatial analysis (§18). In the present case we will see (§18) that the residuals are not independent and a more sophisticated model is needed to get a proper regression equation.

**Random effects model**  In the present example both variables (grain and straw yields) were measured with error; they were not imposed by the experimenter. Both variables should have Gaussian error, with some correlation. This is modelled as a **bivariate normal distribution** of two random variables, \( X \) and \( Y \) with (unknown) population means \( \mu_X \) and \( \mu_Y \), (unknown) population variances \( \sigma_X \) and \( \sigma_Y \), and an (unknown) correlation \( \rho_{XY} \) which is computed as the standardised (unknown) covariance \( \text{Cov}(X, Y) \):

\[
X \sim \mathcal{N}(\mu_X, \sigma_X) \\
Y \sim \mathcal{N}(\mu_Y, \sigma_Y) \\
\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}
\]

This is exactly what was modelled in the previous section.

**Note:** In practice, the distinction between the two models is not always clear. The predictor, even if specified by the experimenter, can also have some measurement error. In the fertiliser experiment, even though we specify the amount per plot, there is error in measuring, transporting, and spreading it. In that sense it can be considered a random variable. But, since we have some control over it, the experimental error can be limited by careful procedures. We can not limit the error in the response by the same techniques.

### 8.2.1 Fitting a regression line – 1

When we decide to consider one of the variables as a response and the other as a predictor, we attempt to fit a line that best describes this relation. There are three types of lines we can fit, usually in this order:

1. Exploratory, non-parametric
2. Parametric
3. Robust

The first kind just gives a “smooth” impression of the relation. The second fits according to some optimality criterion; the classic least-squares estimate is in this class. The third is also parametric but optimises some criterion that protects against a few unusual data values in favour of the majority of the data.

In the present case, our analysis above provides evidence that the relation is indeed well-modelled by a bivariate normal distribution, so that a parametric approach can be taken. We will then examine regression diagnostics to see if a robust regression is needed; these methods are discussed below in §11.

Q37: Is there any reason to choose grain or straw as the dependent variable (predictand), and other as the independent (predictor)? Jump to A37 •

The simplest way to model a parametric linear regression is by ordinary least-squares (OLS). The next section explains the mathematical justification; the computation continues in §8.2.3.

8.2.2 Ordinary least-squares*

Note: This explanation is adapted from Lark and Cullis [27, Appendix] and Draper and Smith [10, §2.6].

In the general linear model, with any number of predictors, there is a \( n \times p \) design matrix of predictor values usually written as \( X \), with one row per observation (data point), i.e., \( n \) rows, and one column per predictor, i.e., \( p \) columns. In the single-predictor with intercept case, it is a \( n \times 2 \) matrix with two columns: (1) a column of 1 representing the intercept, and (2) a column of predictor values \( x_i \). The predictand (response variable) is a \( n \times 1 \) column vector \( y \), one row per observation. The coefficient vector \( \beta \) is a \( p \times 1 \) column vector, i.e., one row per predictor (here, 2). This multiplies the design matrix to produce the response:

\[
y = X\beta + \epsilon
\]

where \( \epsilon \) is a \( n \times 1 \) column vector of residuals, also called errors, i.e., the lack of fit. We know the values in the predictor matrix \( X \) and the response vector \( y \) from our observations, so the task is to find the optimum values of the coefficients vector \( \beta \).

To solve this we need an optimization criterion. The obvious criterion is to minimize the total error (lack of fit) as some function of \( \epsilon = y - X\beta \); the goodness-of-fit is then measured by the size of this error. A common way to measure the total error is by the sum of vector norms; in the simplest case the Euclidean distance from the expected value, which we take to be 0 in order to have an unbiased estimate. If we decide that both positive and negative residuals are equally important, and that larger errors are more

---

9 The dimensions of the matrix multiplication are \( n \times 1 = (n \times p)(p \times 1) \)
serious than smaller, the vector norm is expressed as the sum of squared errors, which in matrix algebra can be written as:

$$S = (y - X\beta)^T(y - X\beta) \quad (8.4)$$

which expands to

$$S = y^T y - \beta^T X^T y - y^T X\beta + \beta^T X^T X\beta$$

$$S = y^T y - 2\beta^T X^T y + \beta^T X^T X\beta \quad (8.5)$$

Note: $y^T X\beta$ is a $1 \times 1$ matrix, i.e., a scalar\(^{10}\), so it is equivalent to its transpose: $y^T X\beta = [y^T X\beta]^T = \beta^T X^T y$. So we can collected the two identical $1 \times 1$ matrices (scalars) into one term.

This is minimized by finding the partial derivative with respect the unknown coefficients $\beta$, setting this equal to 0, and solving:

$$\frac{\partial}{\partial \beta^T} S = -2X^T y + 2X^T X\beta$$

$$0 = -X^T y + X^T X\beta$$

$$(X^T X)\beta = X^T y$$

$$(X^T X)^{-1}(X^T X)\beta = (X^T X)^{-1}X^T y$$

$$\hat{\beta}_{OLS} = (X^T X)^{-1}X^T y \quad (8.6)$$

which is the usual OLS solution.

The above solution depends on an important assumption: the errors must be identically and independently distributed (abbreviated i.i.d.). We did not consider the direction of the error, i.e., with which $y_i$ a particular $\varepsilon_i$ is associated; all errors are considered to be drawn from the same population. This assumption may not be tenable; we will return to this in §18.

8.2.3 Fitting a regression line – 2

**Task 33**: Fit a least-squares prediction line of straw as predicted by grain; display the model summary.

**Q38**: What is the purpose of fitting this relation? Jump to A38

For this we use the **linear models** function `lm`; this is the workhorse of modelling in R. The generic **summary** function now produces a model summary; the **coefficients** access function extracts the regression line coefficients from the model object.

```r
> model.straw.grain <- lm(straw ~ grain, data = mhw)
> summary(model.straw.grain)
```

\(^{10}\) The dimensions of the matrix multiplication are $(1 \times n)(n \times p)(p \times 1)$
Call:
`lm(formula = straw ~ grain, data = mhw)`

Residuals:
```
  Min   1Q Median   3Q  Max
-2.0223 -0.3529  0.0104  0.3734  3.0342
```

Coefficients:
```
          Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8663    0.2387   3.63  0.00031 ***
grain        1.4305    0.0601  23.82 < 2e-16 ***
```

---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.615 on 498 degrees of freedom
Multiple R-squared: 0.533,    Adjusted R-squared: 0.532
F-statistic: 567 on 1 and 498 DF,  p-value: <2e-16

```r
> coefficients(model.straw.grain)

(Intercept)  grain
       0.86628  1.43050
```

This example illustrates the simplest S model formula, here `straw ~ grain`, which is signalled by the `~` operator, which can be read “is modelled by”. So here the formula `straw ~ grain` can be read “straw yield is modelled by grain yield”. The optional `data` argument to the `lm` function informs the function where to look for the variables named in the model formula. The formula could also have named the fields within the object directly: `lm(mhw$straw ~ mhw$grain)`. It’s usually more convenient to use the `data` argument.

---

**Q39:** What is the linear least-squares relation between straw and grain yields? 

**Q40:** How much straw yield is expected if there is no grain yield? Does this seem realistic?

**Q41:** How much does the straw yield increase for each extra lb. of grain yield?

**Q42:** How much of the variability in straw yield is explained by this relation?

**Task 34:** Display the scatterplot with the best-fit line superimposed.

```r
> plot(straw ~ grain, data=mhw)
```
We will investigate whether the model we have just fit satisfies this criterion.

\[ y = \beta_0 + \beta_1 x + \epsilon \]

We will investigate whether the model we have just fit satisfies this criterion.

**Task 35**: Display a histogram and quantile-quantile plot of the **regression residuals**; summarise these numerically.
> hist(residuals(model.straw.grain),
+ main="Residuals from straw vs. grain linear model")

Residuals from straw vs. grain linear model

> qqnorm(residuals(model.straw.grain),
+ main="Residuals from straw vs. grain linear model")
> qqline(residuals(model.straw.grain))
> summary(residuals(model.straw.grain))

Min. 1st Qu.  Median    Mean  3rd Qu.    Max.
-2.0200  -0.3530  0.0104  0.0000  0.3730  3.0300

Q43 : Do the residuals appear to be normally-distributed? Are they at least symmetrically-distributed?
We can test the normality of the residuals with a Shapiro-Wilks test:

```r
> shapiro.test(residuals(model.straw.grain))

Shapiro-Wilk normality test

data: residuals(model.straw.grain)
W = 0.977, p-value = 3.6e-07
```

Q44: According to the Shapiro-Wilk test, should we reject the null hypothesis of normality for the residuals? Jump to A44

Task 36: Compare the fitted to actual values along a 1:1 line; highlight those that are more than 1 lb. plot in error.

To colour points we use the `col` optional argument to the `plot` function, and the `ifelse` function to select a colour based on a logical condition:

```r
> lim <- c(min(fitted(model.straw.grain), mhw$straw),
> max(fitted(model.straw.grain), mhw$straw))
> plot(fitted(model.straw.grain), mhw$straw,
+ xlab="Modelled", ylab="Actual", asp=1,
+ xlim=lim, ylim=lim, pch=20,
+ col=ifelse(
+ (abs(fitted(model.straw.grain) - mhw$straw) < 1),
+ "gray",
+ ifelse(fitted(model.straw.grain) < mhw$straw, "blue","red")),
+ cex=ifelse(
+ (abs(fitted(model.straw.grain) - mhw$straw) < 1),1,1.3)
+ )
> title("Actual vs. modelled straw yields")
> grid()
> abline(0,1)
> rm(lim)
```
**Task 37**: Some of the plots are poorly-fit by this relation; identify those with an absolute residual $> 1.8$ lb. straw and display their records, sorted by grain/straw ratio.

This example illustrates the `abs` ("absolute value"), `which` ("which record numbers?")`, `sort` ("sort values") and `order` ("order records") functions.

We first identify the records with the high absolute residuals, and show them:

```r
> which.high.res <- which(abs(residuals(model.straw.grain)) > 1.8)
> sort(residuals(model.straw.grain)[which.high.res])
```

```
184 35 295 337 311 15
-2.0223 -1.9891 2.3780 2.4107 2.4137 3.0342
```

Now we show these records in the data frame:

```r
> high.res <- mhw[which.high.res, ]
> high.res[order(high.res$grs), ]
```

```
 r  c grain straw  gsr
15 15 1  3.46   8.85 0.39096
337 17 17  3.05   7.64 0.39921
311 11 16  3.74   8.63 0.43337
295 15 15  3.73   8.58 0.43473
184  4 10  4.59   5.41 0.84843
 35 15  2  4.42   5.20 0.85000
```

```r
> rm(which.high.res, high.res)
```

**Q45**: Which plots have absolute residuals $> 1.8$ lb. straw? Which are too high and which too low, according to the relation?  

*Jump to A45*
**Challenge:** Repeat the analysis, but reversing the variables: model grain yield as a function of straw yield. Are the slopes inverses of each other? Why or why not?

**Further diagnostics** The normality of residuals is one requirement for linear modelling; however there are other issues.

The generic `plot` function applied to a model result (i.e. object returned from a call to `lm`) gives a standard set of diagnostic plots, selectable with the `which` argument.

Plot type 1 is a plot of residuals vs. fitted values; there should be no relation between these. That is, whether a residual is high or low, positive or negative should not depend on the value fitted by the model. There should not be any trend; the smooth curve (fit by the `lowess` function) gives a visual impression of this – it should ideally be a horizontal line at 0.

This plot type also helps evaluate whether the variance of the residuals is constant across the range of the predictor, i.e., are they *homoscedastic* as required for fitting simple linear regression by OLS (Equation 8.1): looking vertically at any fitted value, the spread should be identical.

```r
> plot(model.straw.grain, which = 1)
```

Plot type 2 is the Q-Q plot as we studied in the previous section.

Plot type 5 shows the *leverage* of each observation and its corresponding residual.

Observations that are far from the centroid of the regression line can have a large effect on the estimated slope; they are said to have high *leverage*, by analogy with a physical lever. They are not necessarily in error, but they should be identified and verified; in particular, it is instructive to compare the estimated regression line with and without the high-leverage observations.
The leverage is measured by the *hat value*, which measures the overall influence of a single observation on the predictions. In diagnostic plot type 5 the abscissa (‘x-axis’) is the hat value. Two things are of interest:

- No extreme leverage, compared to other observations;
- Residuals at high-leverage observations should not be too much smaller than for other observations. Note that high-leverage observation “pull” the regression towards better fits, so their residuals are expected to be somewhat below average.

```r
> plot(model.straw.grain, which = 5)
```

Plot type 6 is Cook’s distance vs. leverage. Cook’s distance measures how much the estimated parameter vector \( \hat{\beta} \) shifts if a single observation is omitted. A high Cook’s distance means that the observation has a large influence on the fitted model.

We also specify the number of extreme points to label with the `id.n` optional argument.

```r
> plot(model.straw.grain, which = 6, id.n = 10)
```
This plot also shows contours of absolute standardised residuals, as labelled straight lines through the origin.

Q46: Which observations are flagged by their Cook’s distance as having the most influence on the fit? Are these high-leverage observations? Jump to A46.

We will compare this model to more complex ones in §12, below.

8.2.5 Prediction

Once the regression model is built, it can be used to predict: in this case, given a grain yield, what is the expected straw yield? And, how large or small could it be?

The predict generic method specialized to predict.lm for objects of class lm, such as the model we’ve just built. In addition to the model name, it requires a dataframe in which predict will look for variables with which to predict (in this case, a field named grain). In addition, predict.lm can return the confidence interval of the prediction, either of the fit itself or of values predicted from the equation.

There are two sources of prediction error:

1. The uncertainty of fitting the best regression line from the available data;

2. The uncertainty in the prediction, even with a perfect regression line, because of uncertainty in the process which is revealed by the regression (i.e. the inherent noise in the process) These correspond to the confidence interval and the prediction interval, respectively, both at
some level of risk of a Type I error, i.e., that the true value is not in the given range. Clearly, the second must be wider than the first.

The interpretation of these two intervals is as follows:

confidence: We are confident, with only a specified probability of being wrong, that the expected value of the response at the given value of the predictand is within this interval. In this case, if we sampled many plots with the same grain yield, this is the interval where the true mean value lies.

prediction: We are confident, with only a specified probability of being wrong, that any single value of the response at the given value of the predictand is within this interval. In this case, any one plot with the given grain yield should have a straw yield in this interval.

The estimation variance depends on (1) the variance of the regression \( s_{Y,X}^2 \) and (2) the distance \( (x_0 - \bar{x}) \) of the predictand at value \( x_0 \) from the centroid of the regression, \( \bar{x} \). The further from the centroid, the more any error in estimating the slope of the line will affect the prediction:

\[
s_{Y_0}^2 = s_{Y,X}^2 \left[ 1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] \tag{8.7}
\]

This shows that if we try to predict “too far” from the centroid, the uncertainty will be so large that any prediction is meaningless.

Note: The variance of the regression \( s_{Y,X}^2 \) is computed from the deviations of actual and estimated values at all the known points:

\[
s_{Y,X}^2 = \frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \tag{8.8}
\]

\[\text{Task 38} \quad \text{Compute the most likely value, and the 95% confidence and prediction intervals, of straw yields predicted by a grain yield of the mean, and the mean ± one and two standard deviations of the predictor.}\]

We first build a (very) small dataframe with the data.frame function, with a variable of the same name, and then use it as the newdata argument to predict.lm; the other arguments are the model name, the type of interval, and the risk of Type I error.

\[
\begin{align*}
t &\leftarrow \text{mean(mhw$grain)}; \quad t2 \leftarrow \text{sd(mhw$grain)}; \\
p.frame &\leftarrow \text{data.frame(grain=seq(t1-2*t2, t1+2*t2, t2))} \\
predict(model.straw.grain, newdata=p.frame, \\
& \quad \text{interval="confidence", level=0.95}) \\
\text{fit lwr upr} \\
1 5.2037 5.0828 5.3245 \\
2 5.8592 5.7828 5.9357 \\
3 6.5148 6.4608 6.5688 \\
4 7.1704 7.0939 7.2468 \\
5 7.8259 7.7051 7.9468 \\
predict(model.straw.grain, newdata=p.frame, \\
& \quad \text{interval="prediction", level=0.95}) \\
\end{align*}
\]

55
Q47: Which interval is wider? What happens to the width of the interval as the predictand is further from its centroid (mean)?  

Task 39: Display the scatterplot of straw vs. grain yields, with the best-fit line and the two confidence intervals, for grain yields from 2 to 6 lb. acre⁻¹, at 0.1 lb. resolution.

The `seq` function builds the required sequence, and `data.frame` is again used to build a prediction frame. The `plot` function initiates the plot, and then the `title`, `grid`, `lines`, `points`, and `abline` functions add graphic or text elements.

```r
> p.frame <- data.frame(grain = seq(from = 2, to = 6, by = 0.1))
> pred.c <- predict(model.straw.grain, newdata = p.frame,
+ interval = "confidence", level = 0.95)
> pred.p <- predict(model.straw.grain, newdata = p.frame,
+ interval = "prediction", level = 0.95)
> plot(straw ~ grain, data = mhw, pch = 20)
> title(main = "Straw yield predicted by grain yield",
+ sub = "Prediction (blue) and confidence (red) intervals")
> abline(model.straw.grain)
> grid()
> lines(p.frame$grain, pred.c[, "lwr"], col = 2, lwd = 1.5)
> lines(p.frame$grain, pred.c[, "upr"], col = 2, lwd = 1.5)
> lines(p.frame$grain, pred.p[, "lwr"], col = 4, lwd = 1.5)
> lines(p.frame$grain, pred.p[, "upr"], col = 4, lwd = 1.5)
> points(mean(mhw$grain), mean(mhw$straw), pch = 23, cex = 2,
+ bg = "red")
> abline(h = mean(mhw$straw), lty = 2)
> abline(v = mean(mhw$grain), lty = 2)
```
Q48: Why is the confidence interval so narrow and the prediction interval so wide, for this relation?  

Task 40: Clean up from this subsection.

\[ \text{Call: lm(formula = grain ~ straw, data = mhw)} \]

Task 41: Compare the regression of straw yield on grain yield, with the regression of grain yield on straw yield.

\[ \text{Call: lm(formula = grain ~ straw, data = mhw)} \]

Coefficients:
Q49: Is the amount of variability explained by the two models the same? That is, does knowing straw yield give the same amount of information on grain yield as the reverse? Jump to A49

Intuitively it might seem that the slope of grain vs. straw would be the inverse of the slope of straw vs. grain. Is this the case?

Task 42: Compute the slope of straw vs. grain as the inverse of the modelled slope of grain vs. straw, and compare with the modelled slope of straw vs. grain.

> coefficients(model.straw.grain)["grain"]

grain
1.4305

> 1/coefficients(model.grain.straw)["straw"]

straw
2.686

We can visualise these on the scatterplot of straw vs. grain. The regression line of straw on grain can be directly plotted with abline on the model...
object; the reverse regression must be inverted from coefficients extracted from its model object. The slope is just the inverse; the intercept is the straw yield corresponding to zero grain yield:

\[
\begin{align*}
\text{grain} &= b_0 + b_1 \cdot \text{straw} \\
0 &= \text{grain} \\
\downarrow \\
0 &= b_0 + b_1 \cdot \text{straw} \\
\downarrow \\
\text{straw} &= -\frac{b_0}{b_1}
\end{align*}
\]

Program:

```r
> plot(straw ~ grain, pch = 1, main = "Mercer-Hall wheat yields", +     xlab = "grain (lb. plot-1)", ylab = "straw (lb. plot-1)", +     data = mhw)
> title(sub = "straw vs. grain: solid; grain vs. straw: dashed")
> abline(model.straw.grain)
> abline(-beta["(Intercept)"] / beta["straw"], 1 / beta["straw"], +     lty = 2)
> grid()
> rm(beta)
```

Q50: Do these two models give the same straw vs. grain relation? Why not?

Jump to A50 *

So, the regression of two variables on each other depends on which variables is considered the predictor and which the predictand. If we are predicting,
this makes sense: we get the best possible prediction. But sometimes we
are interested not in prediction, but in understanding a relation between
two variables. In the present example, we may ask what is the true relation
between straw and grain in this wheat variety? Here we assume that this
relation has a common cause, i.e. plant growth processes affect the grain and
straw in some systematic way, so that there is a consistent relation between
them. This so-called structural analysis is explained in detail by Sprent [47]
and more briefly by Webster [52] and Davis [8, pp. 218–219].

In structural analysis we are trying to establish the best estimate for a structural
or law-like relation, i.e. where we hypothesise that \( y = \alpha + \beta x \), where
both \( x \) and \( y \) are mathematical variables. This is appropriate when there
is no need to predict, but rather to understand. This depends on the prior
assumption of a true linear relation, of which we have a noisy sample.

\[
X = x + \xi \\
Y = y + \eta
\] (8.9)

That is, we want to observe \( X \) and \( Y \), but instead we observe \( x \) with random
error \( \xi \) and \( y \) with random error \( \eta \). These errors have (unknown) variances \( \sigma_\xi^2 \) and \( \sigma_\eta^2 \), respectively; the ratio of these is crucial to the analysis, and is
symbolised as \( \lambda \):

\[
\lambda = \frac{\sigma_\eta^2}{\sigma_\xi^2}
\] (8.11)

Then the maximum-likelihood estimator of the slope \( \hat{\beta}_{Y,X} \), taking \( Y \) as the
predictand for convention, is:

\[
\hat{\beta}_{Y,X} = \frac{1}{2s_{XY}} \left\{ (s_Y^2 - \lambda s_X^2) + \sqrt{(s_Y^2 - \lambda s_X^2)^2 + 4\lambda s_X^2 s_{XY}} \right\}
\] (8.12)

Equation 8.12 is only valid if we can assume that the errors in the two variables are uncorrelated. In the present example, it means that a large
random deviation for a particular sample of the observed straw yield from
its “true” value does not imply anything about the random deviation of the
observed grain yield from its “true” value.

The problem is that we don’t have any way of knowing the true error variance ratio \( \lambda \) (Equation 8.11), just as we have no way of knowing the true population variances, covariance, or parameters of the structural relation.

We estimate the population variances \( \sigma_X^2 \), \( \sigma_Y^2 \) and covariance \( \sigma_{XY} \) from the
sample variances \( s_X^2 \), \( s_Y^2 \) and covariance \( s_{XY} \), but there is nothing we’ve measured from which we can estimate the error variances or their ratio. However, there are several plausible methods to estimate the ratio:

- If we can assume that the two error variances are equal, \( \lambda = 1 \). This
  may be a reasonable assumption if the variables measure the same
  property (e.g. both measure clay content in different soil layers), use
  the same method for sampling and analysis, and there is an a priori
  reason to expect them to have similar variability (heterogeneity among
  samples). However in this case there is no reason to expect equal
  variances.
The two error variances may be estimated by the ratio of the sample variances: \( \lambda \approx \frac{s^2_y}{s^2_z} \). That is, we assume that the ratio of variability in the measured variable is also the ratio of variability in their errors. For example, if the set of straw yields in a sample is twice as variable as the set of grain yields in the same sample, we would infer that the error variance of straw yields is also twice as much that for grain yields, so that \( \lambda = 2 \). But, these are two completely different concepts! One is a sample variance and the other the variance of the error in some random process. However, this ratio at least normalizes for different units of measure and for different process intensities. Using this value of \( \lambda \) computes the Reduced Major Axis (RMA), which is popular in biometrics.

The variance ratio may be known from previous studies.

---

**Task 43**: Compute the variance ratio of straw and grain yields.

```r
> var(mhw$straw)/var(mhw$grain)
[1] 3.8423
```

**Q51**: Is straw or grain yield more variable across the 500 plots? What is the ratio? *Jump to A51*  

8.3.1 A user-defined function

**Task 44**: Write an R function to compute \( \hat{\beta}_{Y.X} \), given the two structural variables in the order predictand, predictor and the ratio of the error variances \( \lambda \).

This gives us an opportunity to see how to write a user-defined function in the S language. A function has:

1. a **name**, like any R object; this is written at the left of the `<-` assignment operator;
2. the command **function**;
3. a list of named **arguments** immediately following the function name, written within matched parentheses ( ); if there are more than one argument, these are separated by commas (,);
4. the function **body** between matched braces { and }; this is R code which can refer to the named arguments and any other object defined in the workspace at the time the function is called;
5. an optional **return** command, whose argument is evaluated and returned as the value of the function; if no **return** command is given the
value at the end of the function is returned.

The function command creates a function, and we know it must have three arguments: the two structural variables and the ratio of the error variances. We can name the arguments as we wish. Here we choose $y$ and $x$ for the variables, and $\lambda$ for the ratio, and refer to these names in the body of the function. We also have to choose a name for the function object; here we choose a meaningful name, struct.beta.

```r
> struct.beta <- function(y, x, lambda) {
+   a <- var(y) - lambda * var(x);
+   c <- var(x, y);
+   return((a + sqrt(a^2 + 4 * lambda * c^2))/(2*c))
+ }
```

This function is now defined in the workspace and available to be called with the required three arguments.

**Task 45:** Apply this function to the straw vs. grain yields:

1. with $\lambda = 1$; this is the **orthogonal** estimate;
2. with $\lambda$ as the variance ratio of straw and grain yields (assuming the error variance ratio equals the variables' variance ratio); this is the **proportional** estimate.

Compare with the slopes of the forward and reverse regressions.

```r
> print(paste("Forward:", round(coefficients(model.straw.grain)["grain"], 4)))
[1] "Forward: 1.4305"
> print(paste("Proportional:", round(struct.beta(mhw$straw, + mhw$grain, var(mhw$straw)/var(mhw$grain)), 4)))
[1] "Proportional: 1.9602"
> print(paste("Inverse proportional:", round(1/struct.beta(mhw$grain, + mhw$straw, var(mhw$grain)/var(mhw$straw)), 4)))
[1] "Inverse proportional: 1.9602"
> print(paste("Orthogonal:", round(struct.beta(mhw$straw, + mhw$grain, 1), 4)))
[1] "Orthogonal: 2.4031"
> print(paste("Inverse orthogonal:", round(1/struct.beta(mhw$grain, + mhw$straw, 1), 4)))
[1] "Inverse orthogonal: 2.4031"
> print(paste("Reverse:", round(1/coefficients(model.grain.straw)["straw"], + 4)))
[1] "Reverse: 2.686"
```
Note that all the estimates made with the `struct.beta` function are numerically between the slopes of the forward and inverse regressions, which can be considered the extremes (where all error is attributed to one or the other variable).

**Task 46**: Plot the forward, reverse, orthogonal and proportional regression lines on one scatterplot of straw vs. grain yield.

For the models fit with `lm` we can extract the coefficients; for the structural models we compute the slopes with our user-written function `struct.beta` and then their intercepts with the relation:

\[
\hat{\beta}_0 = \hat{\mu}_y - \hat{\beta}_{y,x} \hat{\mu}_x \tag{8.13}
\]

```r
> plot(straw ~ grain, main="Mercer-Hall wheat yields", + sub="Regression slopes", xlab="grain (lb. plot-1)", + ylab="straw (lb. plot-1)", data=mhw)
> abline(model.straw.grain, col="blue")
> beta <- coefficients(model.grain.straw)
> abline(-beta["(Intercept)"]/beta["straw"], 1/beta["straw"], + lty=2, col="green")
> beta <- struct.beta(mhw$straw, mhw$grain, 1)
> abline(mean(mhw$straw)-beta*mean(mhw$grain), beta, lty=3, col="red")
> beta <- struct.beta(mhw$straw, mhw$grain, var(mhw$straw)/var(mhw$grain))
> abline(mean(mhw$straw)-beta*mean(mhw$grain), beta, lty=4, col="brown")
> lines(c(4,4.5),c(5,5), lty=1, col="blue")
> lines(c(4,4.5),c(4.4,4.4), lty=4, col="brown")
> lines(c(4,4.5),c(4.6,4.6), lty=3, col="red")
> lines(c(4,4.5),c(4.8,4.8), lty=2, col="green")
> grid()
> text(4.5,5,paste("Forward:", + round(coefficients(model.straw.grain)["grain"],4)), + col="blue", pos=4)
> text(4.5,4.4,paste("Proportional:", + round(struct.beta(mhw$straw,mhw$grain,var(mhw$straw)/var(mhw$grain)),4)), + col="brown", pos=4)
> text(4.5,4.6, paste("Orthogonal:", + round(struct.beta(mhw$straw,mhw$grain,1),4)), + col="red", pos=4)
> text(4.5,4.8,paste("Reverse:", + round(1/coefficients(model.grain.straw)["straw"],4)), + col="green", pos=4)
```
**Q52**: What do you conclude is the best numerical expression of the structural relation between straw and grain yields in this variety of wheat, grown in this field?  

**Challenge**: Modify function `struct.beta` to return both the intercept and slope of the structural line\(^{11}\), and use this to simplify the display of lines on the scatterplot.

### 8.4 No-intercept model\(^*\)

In the simple linear regression of §8.2 the model is:

\[
y_i = \beta_0 + \beta_1 x_i + \epsilon_i
\]  

(8.14)

This has two parameters: the slope $\beta_1$ and the intercept (value of the predictand when the predictor is 0) $\beta_0$. It is also possible to fit the model without an intercept, i.e., the linear relation is forced through the origin $(0, 0)$. The equation becomes:

\[
y_i = \beta x_i + \epsilon_i
\]  

(8.15)

There is then only a slope to be estimated, since the intercept is fixed at 0. These are termed **no-intercept** models.

**Q53**: Why might this have some logic in the case of predicting straw yield from grain yield?  

\(^{11}\) Hint: use the c “make a list” function
There are some mathematical implications of a no-intercept model.

- The mean residual is (in general) not zero;
- The residual sum-of-squares is (in general) larger than for a model with intercept;
- The usual formula for goodness-of-fit is not appropriate (§8.4.2).

Even if we know from nature that the relation must include \((0, 0)\), this takes away a degree of freedom from the fit, and gives a poorer fit in the range of observed responses, if this does not include \(y = 0\).

A no-intercept model may be appropriate when:

1. There are physical reasons why the relation must include \((0, 0)\); e.g., no straw → no grain is possible (but not vice-versa!);
2. If a negative prediction should be avoided; e.g., it is impossible to have negative straw or grain in a plot\(^{12}\);
3. If the range of the observations covers or approaches \((0, 0)\); otherwise we are assuming a linear form from the origin to the range of our data, when it may have some other form, e.g., exponential, power . . . ; there is no evidence for choosing a linear form near the origin;
4. If, after fitting a with-intercept model, the null hypothesis \(H_0 : \beta_0 = 0\) in a linear regression with intercept can not be disproven (\(t\)-test of the coefficient), and we want to simplify the relation, we may then choose to re-fit with a no-intercept model.

### 8.4.1 Fitting a no-intercept model

In a no-intercept model, the slope \(\hat{\beta}_{Y,X}\) can not be estimated from the sample covariance \(s_{XY}\) and variance of the predictand \(s_x^2\), because the (co)variances are relative to means, which we can not compute; this is because the fixed intercept removes this degree of freedom.

Instead, the slope is computed by minimizing the RSS, again by orthogonal projection: \(b = (x'x)^{-1}x'y\), where the design matrix \(x\) here does not have an initial column of 1’s, just a column of \(x_i\). In the univariate case this reduces to \(\sum x_i y_i / \sum x_i^2\).

---

**Task 47**: Fit a no-intercept model of straw yield predicted by grain yield and summarize it.

In the R model formulas, absence of the intercept is symbolized by the term \(-1\) in the formula supplied to the `lm` function:

```r
> model.straw.grain.0 <- lm(straw ~ grain - 1, data = mhw)
> summary(model.straw.grain.0)
```

\(^{12}\)But this can also be avoided by setting any negative predictions to zero.
Call:
\texttt{lm(formula = straw ~ grain - 1, data = mhw)}

Residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2.1496</td>
<td>-0.3660</td>
<td>0.0292</td>
<td>0.3657</td>
<td>3.1515</td>
</tr>
</tbody>
</table>

Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| grain    | 1.647      | 0.007   | 235      | <2e-16 *** |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.622 on 499 degrees of freedom
Multiple R-squared: 0.991, Adjusted R-squared: 0.991
F-statistic: 5.54e+04 on 1 and 499 DF, p-value: <2e-16

Q54: What is the slope of the relation? Does this differ from the $\beta_1$ coefficient of the with-intercept model? Why? Jump to A54

Task 48: Display a scatterplot of the straw vs. grain yields, with the with- and no-intercept lines superimposed. Show the origin (0, 0). Also show the centroid of the points. We take this opportunity to introduce an efficient way to refer to field names within a data frame, without having to name the frame each time a field is named. So, instead of writing \texttt{mhw$grain} anywhere in an expression, we can just write \texttt{grain}. This is the \texttt{with} function, that evaluates its second argument (an R expression) while exposing any names within the object named as its first argument. Here writing \texttt{with(mhw, ...} exposes the field names such as \texttt{grain} to be used in the expression.

As a simple example, the following are equivalent:

\begin{verbatim}
> mean(mhw$grain)
[1] 3.9486
> with(mhw, mean(grain))
[1] 3.9486
\end{verbatim}

We use this here to simplify the arguments to the following \texttt{plot} command:

\begin{verbatim}
> with(mhw, 
  + plot(straw ~ grain, main="Mercer-Hall wheat yields", 
  + xlab="grain (lb. plot-1)", ylab="straw (lb. plot-1)", 
  + xlim=c(0,ceiling(max(grain))), 
  + ylim=c(0, ceiling(max(straw))), cex=0.8)) 
> abline(model.straw.grain, col="blue") 
> abline(model.straw.grain.0, col="red") 
> grid() 
> text(4.5,4, paste(" With:",
\end{verbatim}
Mercer–Hall wheat yields

Task 49: Confirm that the mean residual of the no-intercept model is not zero, whereas that for the with-intercept model is.

```r
> mean(residuals(model.straw.grain.0))
[1] 0.011491
> mean(residuals(model.straw.grain))
[1] 1.7643e-16
```

Q55: Is the no-intercept model appropriate in this case? Jump to A55
8.4.2 Goodness-of-fit of the no-intercept model

The coefficient of determination $R^2$ for no-intercept model is in general not a good measure of fit, and is usually massively inflated, for the following reason.

Since there is no intercept in the design matrix, the total sum of squares (TSS) must be computed relative to zero: $TSS = \sum_{i=1}^{n}(y_i - 0)^2$, rather than relative to the sample mean $\bar{y}$: $TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2$. We still define $R^2$ as:

\[
R^2 = 1 - \frac{RSS}{TSS}
\]

But since the TSS is computed relative to zero, it tends to be quite high (no compensation for the sample mean), so even though the residual sum of squares (RSS) is larger than if an intercept is included, the $R^2$ tends to be very high.

**Task 50**: Compute the coefficient of determination $R^2$ and the root mean squared error for the no- and with-intercept models.

First, we compute $R^2$ directly from the definitions:

```r
> (TSS <- sum((mhw$straw - mean(mhw$straw))^2))
[1] 402.67
> (TSS0 <- sum(mhw$straw^2))
[1] 21624
> (RSS <- sum(residuals(model.straw.grain)^2))
[1] 188.22
> (RSS0 <- sum(residuals(model.straw.grain.0)^2))
[1] 193.19
> (R2 <- 1 - (RSS/TSS))
[1] 0.53258
> (R20 <- 1 - (RSS0/TSS0))
[1] 0.99107
> rm(TSS, TSS0, RSS, RSS0, R2, R20)
```

Notice how the total sums of squares is much higher for the no-intercept model, because it is relative to 0 rather than the sample mean. The residual sum of squares is a bit higher, because the fit through the points is not so close when an intercept is not allowed; however, in this case (and in general) the RSS is only a bit higher.
Second, we show the $R^2$ computed with the model; that this is adjusted for the number of model parameters and sample size.

```r
> summary(model.straw.grain.0)$adj.r.squared
[1] 0.99105

> summary(model.straw.grain)$adj.r.squared
[1] 0.53164
```

Q56: (1) What is the relation between the adjusted and raw $R^2$ for both models? Compare the amount of adjustment; are they the same? Why not?

(2) What happens to the $R^2$ in this case when the intercept is removed from the model? Is this a realistic view of the success of the model? Jump to A56

We also compute the root mean squared error (RMSE), i.e., lack of fit, from the RSS and the number of observations, for both models.

```r
> sqrt(sum(residuals(model.straw.grain)^2)/(length(mhw$straw)))
[1] 0.61354

> sqrt(sum(residuals(model.straw.grain.0)^2)/(length(mhw$straw)))
[1] 0.6216
```

Q57: What happens to the RMSE when the model is forced through $(0, 0)$? Why? Jump to A57

8.5 Answers

A32: Linear. Return to Q32

A33: The diagonals are the variances of the two variables, both in (lb. plot$^{-1}$) squared; the off-diagonals are the covariances between the two variables, in this case also in (lb. $^{-1}$) squared, because the two variables have the same units of measure. Return to Q33

A34: The summary statistics are quite similar, for both variables; the simulation reproduces the statistics of the actual data. Return to Q34

A35: The relations look quite similar; this supports the hypothesis. However,
the bivariate simulation seems to have a slightly steeper slope than the actual data.

A36: The most probable value is 0.73 lb. plot\(^{-1}\); the lower and upper confidence limits are 0.686 and 0.768 lb. plot\(^{-1}\), respectively.

Assessment of the strength is subjective and depends on the application field; the author would call this a moderate positive correlation.

A37: No; both are caused by the same underlying process (plant growth in response to the environment), and neither is more under control of the experimenter. However, see the next question.

A38: It can be used to understand plant physiology: is grain yield a direct result of straw yield, or at least do large plants (lots of straw) tend to have high yield (lots of grain)? Practically, we could use this relation to predict straw yield on other plots where only grain was weighed; it is much easier to collect and weigh grain than straw.

A39: straw = 0.866 + 1.43 \cdot grain

A40: 0.866 lb.; maybe wheat this small would indeed have no grain, because the plants would be too weak to form grain.

A41: 1.43 lb. of straw increase for each lb. of grain increase.

A42: 53.2\% (the value of the adjusted R\(^2\)).

A43: It’s clear that the highest residuals are too low and vice-versa; the histogram is somewhat peaked. The median residual is slightly biased (−0.01). The range is quite high, from −2 to +3 lb. plot\(^{-1}\).

A44: The p-value (probability that rejecting the null hypothesis would be an error) is almost zero, so we should reject the null hypothesis: these residuals are not normally-distributed. This is due to the deviations at both tails.

A45: Plots 15, 337, 311 and 295 have very low grain/straw ratios, so the linear relation predicts too much grain; for plots 35 and 184 it’s the reverse.

A46: Observations 292, 184, 15, and especially 337. Plots 337 and 292 also have high leverage. In the previous answer we saw that plots 15 and 337 have very low grain/straw ratios, so the linear relation predicts too much grain; for plot 184 it’s
the reverse. Plot 292 has high leverage and fairly high Cook’s distance, but its standardised residual is not so high (< 2).

Return to Q46

A47 : The prediction intervals are much wider at all values of the predictand. Intervals further away from the centroid are increasingly wide.

Return to Q47

A48 : The confidence interval is narrow because the average linear relation is very consistent across its range (although, see §18 for some exceptions at the extreme values), so the estimate of the best-fit line is quite good. The prediction interval is wide because there is poor correlation in the sample set, i.e., a wide spread in straw yields for any observed grain yield. So this same uncertainty must appear in the prediction.

Return to Q48

A49 : The models explain the same proportion of the total variability, 0.532.

Return to Q49

A50 : The slopes are very different. The forward model gives a slope of 1.43 of straw vs. grain, whereas the inverse model gives a slope of 2.686. The reason is that the two regressions minimise different error sums-of-squares: in the forward model, residual straw yield, in the inverse model, residual grain yield. Each is the best predictive relation for its target variable.

Return to Q50

A51 : The variance ratio is 3.842, that is, straw yields are almost four times as variable as grain yields. This is partly explained by the higher absolute values of all the yields: the medians are 6.36 for straw vs. 3.94 for grain.

Return to Q51

A52 : The best estimate of the error variance ratio is the variable variance ratio, so the structural relation is 1.96 lb. straw for each lb. grain; or equivalently 0.51 lb. grain for each lb. straw. This is the best estimate of the plant morphology.

Return to Q52

A53 : Physically, if there is no grain, there is no straw. Thus the point (0, 0) is by definition part of the straw vs. grain response relation.

Return to Q53

A54 : The slope is 1.65, considerably steeper than the slope of the with-intercept model, 1.43. This compensates for the intercept (here, forced to 0) being smaller than the fitted intercept of the full model, 0.87, which allows the line to have a shallower slope while passing through the centroid.

Return to Q54

A55 : No, for three reasons. (1) The intercept from the full model is highly unlikely to be zero, so the no-intercept model is not appropriate; (2) the range of the observations is far from (0, 0) so there is no reason to guard from negative predictions; (3) we have no evidence for the model form near the origin; the closest
points are around (2.8, 4).

A56: (1) For both models the adjusted $R^2$ is lower than the raw $R^2$, because of the adjustment for the number of parameters used in the model. The difference for the no-intercept model is less, because only one, rather than two, parameters are used in the model.

(2) The adjusted $R^2$ increases from 0.53 to 0.99, i.e., almost 1. This is an artefact of the calculation and does not reflect the success of the no-intercept model.

A57: The RMSE increases from 0.614 (with-intercept) to 0.622 (no-intercept) lb. acre$^{-1}$; this shows that the no-intercept line does not come as close, on average, to the points in the scatterplot. This is because the slope is not free to float at both ends (find the optimum intercept); instead it is forced through (0,0) as one point, and the other point is then the centroid of the point cloud in the scatterplot.

9 Bivariate modelling: continuous response, classified predictor

A continuous variable can also be modelled based on some classified (discrete) predictor. In the present example we will consider the field half (North & South) to be such a predictor.

It has been suggested [49] that the North and South halves of the field had been treated differently prior to Mercer & Hall’s experiment. To test this suggestion, we first have to code each plot according to its location in the field (N or S half); this is a logical variable (True or False, written in S as TRUE or FALSE, abbreviated T and F). Then we use this variable to split the field statistically and compare the yields for each half. Each plot is thus in one or the other class, in this case field half.

Task 51: Add a logical field to the data frame to codes whether each plot falls in the north half or not.

We first use a logical expression that evaluates to either T or F to create a logical variable, here named in.north, as a field in the data frame. This field codes whether teach plot falls in the north half or not.

Recall from the description of the dataset in Appendix A that the rows ran W to E, with 25 plots in each row, beginning at 1 on the W and running to 25 at the E, and the columns run N to S with 20 plots in each, beginning at 1 on the N and running to to 20 at the S. So the N half of the field consists of the plots with row numbers from 1 to 10, inclusive.

> mhw <- cbind(mhw, in.north = (mhw$r < 11))

> str(mhw)

'data.frame': 500 obs. of 6 variables:
$ r : int 1 2 3 4 5 6 7 8 9 10 ...
$ c : int 1 1 1 1 1 1 1 1 1 1 ...
$\text{grain} : \text{num} 3.63 \ 4.07 \ 4.51 \ 3.9 \ 3.63 \ 3.18 \ 3.42 \ 3.97 \ 3.4 \ldots$

$\text{straw} : \text{num} 6.37 \ 6.24 \ 7.05 \ 6.91 \ 5.93 \ 5.59 \ 5.32 \ 5.52 \ 6.03 \ 5.66 \ldots$

$\text{gsr} : \text{num} 0.57 \ 0.652 \ 0.64 \ 0.564 \ 0.612 \ldots$

$\text{in.north}: \text{logi} \ TRUE \ TRUE \ TRUE \ TRUE \ TRUE \ TRUE \ldots$

> summary(mhw$\text{in.north})

<table>
<thead>
<tr>
<th>Mode</th>
<th>FALSE</th>
<th>TRUE</th>
<th>NA's</th>
</tr>
</thead>
<tbody>
<tr>
<td>logical</td>
<td>250</td>
<td>250</td>
<td>0</td>
</tr>
</tbody>
</table>

**Task 52**: Display a post-plot of grain yields with the plots in the North half coloured blue, those in the South coloured grey\(^{13}\).

> with(mhw, plot(c, r, col = ifelse(in.north, “blue”, “darkslategrey”),
> +     cex = 1.3 * straw/max(straw), pch = 1, xlab = ”Column”,
> +     ylab = ”Row”, ylim = c(20, 1), sub = ”North: blue; South: gray”)
> abline(h = 10.5, lty = 2)

*Postplot of straw yields, coded by field half*

*Note*: The *ylim* graphics argument specifies that the labels of the y-axis run from 20 (lower left corner) to 1 (upper left corner); by default scatterplots drawn by the *plot* function assume the lowest-numbered row is the lower left. This is the usual case for scatterplots, but here we know the lowest-numbered row is at the N side.

9.1 Exploratory data analysis

We first compare the two halves with exploratory graphics; a suitable graph is the *boxplot*, created with the *boxplot* function.

**Task 53**: Compare the two field halves with box plots.

\(^{13}\)An obvious historical reference.
To compare these on one graph, we split the graphics frame by specifying the number of rows and columns with the `mfrow` argument to the `par` (“graphics parameters”) command. These plots look better if they are displayed horizontally, using the optional `horizontal` argument with the value `TRUE`. The optional `names` argument labels the plots; these are `S` and `N` to represent the internal values `FALSE` and `TRUE` of the `in.north` classifier.

```r
> par(mfrow=c(3,1))
> boxplot(grain ~ in.north, names=c("S", "N"),
+ main="Grain yield", horizontal=T, data=mhw)
> boxplot(straw ~ in.north, names=c("S", "N"),
+ main="Straw yield", horizontal=T, data=mhw)
> boxplot(gsr ~ in.north, names=c("S", "N"),
+ main="Grain/straw ratio", horizontal=T, data=mhw)
> par(mfrow=c(1,1))
```

**Q58**: *Do the two halves appear to have different ranges? medians? spreads? Comment for all three variables.*  

We then compare the two halves numerically:

**Task 54**: Compare the summary statistics of the two halves for all the variables. Also compare their variances.

Any function can be applied to subsets of a data frame with the `by` function. The first argument is the argument to the function, the second is the subset classifier, and the third the function to be applied:

```r
> with(mhw, by(grain, in.north, summary))
```

```markdown
<table>
<thead>
<tr>
<th>in.north: FALSE</th>
<th>Min.</th>
<th>1st Qu.</th>
<th>Median</th>
<th>Mean</th>
<th>3rd Qu.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
Q59: Do the two halves have different summary statistics? Is one half more variable than the other? Jump to A59

From the boxplots, it appears that the straw yield is, on average, higher in the S half; can we confirm this with a statistical test?
Task 55: Test whether the straw yield is higher in the N half.

There are two approaches that give the same answer for a binomial classified predictor: a two-sample t-test, and a one-way ANOVA.

9.2 Two-sample t-test

The simplest way to do this is with a **two-sample unpaired t test** of the difference between means, with the default **null hypothesis** that they are identical. This only works when the classified variable is binary.

```
> with(mhw, t.test(straw[in.north], straw[!in.north]))

Welch Two Sample t-test

data: straw[in.north] and straw[!in.north]
t = -6.02, df = 498, p-value = 3.5e-09
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
-0.61969 -0.31455
sample estimates:
mean of x mean of y
6.2812 6.7484
```

Q60: **Is there a significant difference in the means? What is the probability that this apparent difference is only by chance (i.e. a Type I error would be committed if we reject the null hypothesis)? What is the 95% confidence interval of the difference?**

Jump to A60

9.3 One-way ANOVA

Another way to analyse this is with a **one-way Analysis of Variance** (ANOVA). This can also deal with multivalued classified predictors, although in this case we only have a binary predictor.

Task 56: Compute a one-way ANOVA for straw yield between field halves.

This illustrates another use of the `lm` function, i.e. modelling a continuous response variable from a categorical (here, binary) predictor:

```
> model.straw.ns <- lm(straw ~ in.north, data = mhw)
> summary(model.straw.ns)

Call:
lm(formula = straw ~ in.north, data = mhw)

Residuals:
  Min     1Q Median     3Q    Max
-2.181  -0.608  -0.108   0.572   2.359
```
Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|)  |
|----------|------------|---------|-----------|
| (Intercept) | 6.7484 | 0.0549 | 122.90 | < 2e-16 *** |
| in.northTRUE | -0.4671 | 0.0777 | -6.02 | 3.5e-09 *** |

---

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.868 on 498 degrees of freedom
Multiple R-squared: 0.0677, Adjusted R-squared: 0.0659
F-statistic: 36.2 on 1 and 498 DF, p-value: 3.48e-09

Q61: How much of the total variability in straw yield is explained by field half?

We can also see the results with a traditional ANOVA table:

```r
> anova(model.straw.ns)
Analysis of Variance Table

Response: straw

         Df Sum Sq Mean Sq F value Pr(>F)
in.north  1   27.28   27.28 36.20 3.5e-09 ***
Residuals 498  375.00   0.75
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

And of course we must check the regression diagnostics:

```r
> qqnorm(residuals(model.straw.ns),
+        main="Residuals from one-way ANOVA",
+        sub="Straw yield vs. field half")
> qqline(residuals(model.straw.ns))
```
Q62: Are the model residuals normally-distributed? What does this imply about the process in the field? Jump to A62

We will compare this model to more complex ones in §12, below.

Challenge: Repeat the analysis of this section, but splitting the field into E-W halves, rather than N-S halves. Do you reach similar conclusions about the differences between the field halves?

9.4 Answers

A58: Grain yields appear to be almost identically distributed, although the S half is slightly more variable. Straw yields appear slightly higher in the S. The grain/straw ratio appears higher in the N. Variability between field halves seems similar for grain and straw, but the grain/straw ratio in the S half appears to have more total spread and boxplot outliers. Return to Q58

A59: Grain: Almost identical summary statistics; Straw: The S is somewhat higher in all summary statistics, but the variability is almost the same; Grain/straw ratio: the N is higher in all summary statistics except the maximum, which is almost the same; the N is also somewhat more variable. Return to Q59

A60: Yes, very highly significant; the N has a higher ratio than the S. The probability of a Type I error is almost 0. The 95% confidence interval is -0.62, -0.315, i.e. there is only a 2.5% chance that the difference in yields is not at least

14 Make sure to pick an appropriate colour scheme for the classified postplot.
0.315. Return to Q60

A61: In the summary, the Adjusted $R^2$ gives this proportion; here it is 0.066, a very low proportion. This shows that a model can be highly significant – the difference between class means is almost surely not zero – but numerically not important. Return to Q61

A62: For the most part yes, but the tails (highest and lowest ratios) are not well-modelled: the absolute residuals are too high at both tails, especially the lower tail. This suggests that the extreme values are caused by some process that is beyond what is causing most of the “random” variability. This is sometimes called a “contaminated” process. Return to Q62

10 Bootstrapping*

Most of the estimates of population statistics based on sample estimates, as computed in the previous sections, rely on assumptions that are difficult to verify.

For example, we saw in §8.1.1 that the parametric correlation coefficient is only justified for the case of a bivariate normal distribution. Although in this case the simulation based on the sample variance-covariance matrix seemed to support this assumption, we did notice several observations well outside the “envelope” expected if the distribution of the two variables is in fact bivariate normal. Also in the univariate modelling of §7 we could see that the distribution of grain yield was not completely normal: the highest yields are not as high, and the lowest not as low, as expected if the yields were normally distributed.

Further, any confidence intervals for both parametric and non-parametric statistics rely on a major assumption: that the sample estimate approaches the true population value asymptotically; that is, as the sample gets larger, the estimate gets more precise in a smooth manner. All the classical confidence intervals depend on this assumption, which by its nature can not be proven. Further, the smaller the sample (typical of many studies), the less the asymptotic assumption is justified.

Another approach has been made possible by the advent of fast computers. This is bootstrapping, first proposed by Efron in the late 1970’s [13]. Suppose we could repeat whatever experiment gave rise to the one dataset we have – in this case, it would be another uniformity trial with the same design as Mercer and Hall’s original trial and under the same conditions (wheat variety, weather, soil, cultural practices . . . ). If we repeated the trial a large number of times, we’d have a direct estimate of the confidence interval: “Out of 1,024 trials, 95% of the correlation coefficients between grain and straw were above 0.67.” This is another way of saying that we have 95% confidence that any future trial would show a correlation that high or higher.

But of course, we can’t repeat most trials many times – either it is too expensive, would take too long, or is logistically impractical – in the present
example we couldn’t reproduce the same weather as in summer 1910 and even in the same field the soil properties could have changed due to the previous trial.

Efron’s idea was to simulate many trials by resampling the existing trial. This is a subtle idea which at first seems like a trick. But, recall that the actual sample in hand is in fact the best non-parametric information about the true distribution in a larger population. In the present case we have 500 valid observations of grain and straw yield.

As a simple example, suppose we want to know the worst-case grain yield of a small plot (as in the experiment), say the yield with only 1% chance that a given yield would be smaller. With our current sample we can simply take the 1% quantile, symbolised as $q_{0.01}$; in this case with 500 observations, this is the mean of the 5th and 6th-smallest values:

```r
> quantile(mhw$grain, p = 0.01, type = 5)
1%
2.945
> mean(sort(mhw$grain)[5:6])
[1] 2.945
```

*Note:* The `type=5` specifies a piecewise linear function through the values; the default `type=7` uses a slightly different computation; see Hyndman and Fan [23] for an extensive discussion of different ways to compute quantiles of a continuous distribution from an empirical distribution.

But of course this is based on a single experiment with 500 observations. What if we could repeat this experiment many times?

Efron proposed a non-parametric bootstrap which uses the sample in hand as the population, and simulates a new sample from this “population” by picking a number of observations with equal probability, with replacement. That is, a given observation can be picked multiple times. Then this simulated sample is used to estimate the statistic of interest. We can do this 100’s or 1000’s of times and then summarise the statistic.

Bootstrapping is nicely introduced, without formulas, by Shalizi [45] and explained in more detail in the texts of Efron [12] and Davison and Hinkley [9]. The boot R package provides functions related to bootstrapping.

The basic function of the `boot` package is also named `boot`. The default is for a non-parametric bootstrap, specified with the optional `sim` argument as `sim="ordinary"` (the default).

Setting up the bootstrap can be quite complicated. The user must write a function that computes the statistic of interest for each bootstrap replicate. This function has a different form depending on the `boot` optional arguments; in this simplest case it has two arguments:

---

15 otherwise we’d just get the original sample
1. the data, typically a data frame;
2. a list of the indices of the rows (observations) selected for the bootstrap.
   These may be repeated; the bootstrap samples with replacement.

10.1 Example: 1% quantile of grain yield

We begin with a simple example: the 1% quantile (i.e., estimate of the lowest grain yield, with 99% being greater). We estimated this from the single sample as:

```
> quantile(mhw$grain, p = 0.01, type = 5)
```

```
1%
2.945
```

The bootstrapped estimate will compute this many times, each with a different simulated sample.

**Task 57**: Write a function to return the statistic “1% quantile”.

We already wrote a user-defined function in §8.3.1. Again, we use the function command to create a function in the workspace. In this case it must have two arguments: the data frame and the selected indices. We choose meaningful names data and indices, and refer to these names in the body of the function. We choose a meaningful name for the function object: boot.q01.

```
> boot.q01 <- function(data, indices) {
+   obs <- data[indices,]
+   return(quantile(obs$grain, p = 0.01, type = 5))
+ }
```

In the function body, the line `obs <- data[indices,]` makes a data frame with rows corresponding to the bootstrap sample, with the same structure as the original frame. So then the function call to `quantile` refers to field grain in the resampled dataframe, which has the same structure as the original frame mhw but a different set of 500 observations from the resampling.

A function typically ends with the return command, which specifies the value to return to the caller; here it’s the statistic of interest, computed on the replicate.

**Task 58**: Estimate the population statistic “1% quantile” with a non-parametric bootstrap.

We call boot with three arguments: the data, the function to compute the statistic on the replicate (i.e., the function we just wrote), and the number of replicates (argument R). Before calling boot, we must first load the optional boot package, by using the require function.
> require(boot)
> b.q01 <- boot(mhw, boot.q01, R = 1024)
> print(b.q01)

ORDINARY NONPARAMETRIC BOOTSTRAP

Call:
  boot(data = mhw, statistic = boot.q01, R = 1024)

Bootstrap Statistics :
    original  bias  std. error
t1* 2.945 -0.0071045 0.072935

The output of boot shows:

original : the statistic (here, the 1% quantile) applied to the original dataset; in
  this example, this is the same as the result of the (non-bootstrapped)
  R command quantile(mhw$grain, p=0.01, type=5);

bias : the average difference of the bootstrapped estimates from the original
  value; this should be zero;

standard error : of the replicated statistic; the lower this is, the more consistent is the
  estimate.

Note that each time that boot iscalled, a random set of replicates is generated,
so the statistics will vary.

The bootstrapped estimates can be summarised graphically with the plot
method; this recognises the object of class boot and produces two plots: a
histogram of the estimate (with the non-bootstrapped estimate shown as a
dashed vertical line) and its normal Q-Q plot.

> plot(b.q01)

Q63 : Describe the histogram of the 1 024 bootstrapped estimates of \( q_{0.01} \),
also with respect to the single (non-bootstrapped) estimate. Jump to A63

Q64: Explain the “discontinuous” form of the histogram and Q-Q plot. Jump to A64

Bootstrapped confidence intervals With these in hand we can compute the confidence intervals of the statistic. This is one of the main applications of bootstrapping.

There are various ways to compute bootstrapped confidence intervals; the two most used are the normal approximation and the basic bootstrapped estimate; see Davison and Hinkley [9, Ch. 5] for a lengthy discussion.

Normal: Assumes that the empirical distribution of the statistic is asymptotic to a normal distribution, so the bias $b$ and standard error $s$ computed from the empirical bootstrap estimates can be used to compute a normal confidence interval for the population statistic $t$:

$$ t - b - s \cdot z_{1-\alpha}, \ t + b + s \cdot z_{\alpha} $$

(10.1)

where $t$ is the statistic of interest and $\alpha$ specifies the $(1-2\alpha)$ interval; e.g., $\alpha = 0.025$ specifies the 0.95 (95%) interval.

Basic: When there is evidence that the empirical distribution of the statistic is not asymptotic to normal (e.g., as revealed by the Q-Q normal probability plot of the estimates $t_{*1}, t_{*2}, \ldots, t_{*n}$), the normal approximation is not justified. Instead, the value of the quantile is extracted directly from the empirical distribution of the statistic.

Task 59: Compute bootstrap estimates of the of the 1% quantile of grain yield and the bootstrap estimate of the normal approximation and basic 95% confidence intervals.

The original estimate of any bootstrap statistic is found in the $t0$ field of the object returned by `boot`; the bootstrap estimates are in field $t$. So to get the best estimate we average the bootstrap estimates in field $t$:

```r
> mean(b.q01$t)
[1] 2.9379
> b.q01$t0
1%
2.945
```

Q65: How does the average bootstrapped estimate of the 1% quantile compare to the estimate from the original sample? Jump to A65
The `boot.ci` function computes confidence intervals; the `conf` argument gives the probability $2\alpha$ and the `type` argument specifies the type of computation.

```r
> (b.q01.ci <- boot.ci(b.q01, conf = 0.95, type = c("norm", + "basic")))
```

BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1024 bootstrap replicates

CALL :
boot.ci(boot.out = b.q01, conf = 0.95, type = c("norm", "basic"))

Intervals :
Level Normal Basic
95%  ( 2.809,  3.095 ) ( 2.840,  3.080 )
Calculations and Intervals on Original Scale

Q66: What are the bootstrapped estimates of the 95% confidence interval for the 1% quantile? Jump to A66 •

Q67: State the basic interval in terms of probability. Jump to A67 •

10.2 Example: structural relation between grain and straw

In §8.3 we investigated the structural relation between grain and straw; this should be an inherent property of the wheat variety.

Recall that the function to compute a structural relation is:

```r
> struct.beta <- function(y, x, lambda) {
+ a <- var(y)-lambda*var(x);
+ c <- var(x,y);
+ return((a + sqrt(a^2 + 4 * lambda * c^2))/(2*c))
+ }
```

where the variance ratio `lambda` partitions the error between the two variables. For the reduced major axis (RMA) structural relation, the ratio of error variances is estimated as the ratio of sample variances; then the structural relation is estimated as a slope and intercept of:

```r
> beta <- struct.beta(straw, grain, var(straw)/var(grain))
> alpha <- mean(straw) - beta * mean(grain)
```

Task 60: Determine the most likely value and 95% confidence interval for the slope and intercept of the structural relation, with a non-parametric bootstrap.

We first use the `function` function (!) to write a function to compute the statistics, naming it `boot.sr`. Since this function calls the function
struct.beta to compute the slope of the structural relation, that function
must be already defined. The newly-defined function returns a list of two
values; the boot function will then record both of these in field t of the boot
object.

> boot.sr <- function (data, indices) {
+ obs <- data[indices,]
+ beta <- struct.beta(obs$straw, obs$grain,
+ var(obs$straw)/var(obs$grain))
+ alpha <- mean(obs$straw)-beta*mean(obs$grain)
+ return(c(beta, alpha))
+ }

Then the bootstrap:

> b.sr <- boot(mhw, boot.sr, R = 1024)
> print(b.sr)

ORDINARY NONPARAMETRIC BOOTSTRAP

Call:
  boot(data = mhw, statistic = boot.sr, R = 1024)

Bootstrap Statistics :

<table>
<thead>
<tr>
<th></th>
<th>original</th>
<th>bias</th>
<th>std. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>t1</td>
<td>1.9602</td>
<td>0.0019097</td>
<td>0.054296</td>
</tr>
<tr>
<td>t2</td>
<td>-1.2252</td>
<td>-0.0083487</td>
<td>0.213551</td>
</tr>
</tbody>
</table>

Visualise the bootstrap; first the slope:

> plot(b.sr, index = 1)

and then the intercept:

> plot(b.sr, index = 2)
Finally, from this the normal and basic confidence intervals, along with the mean (best estimate):

```r
> mean(b.sr$t[, 1])
[1] 1.9621

> (b.sr.ci <- boot.ci(b.sr, conf = 0.95, type = c("norm", + "basic"), index = 1))

BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1024 bootstrap replicates

CALL :
boot.ci(boot.out = b.sr, conf = 0.95, type = c("norm", "basic"),
       index = 1)

Intervals :
Level Normal  Basic
95%  ( 1.852,  2.065 ) ( 1.849,  2.067 )

Calculations and Intervals on Original Scale

> mean(b.sr$t[, 2])
[1] -1.2336

> (b.sr.ci <- boot.ci(b.sr, conf = 0.95, type = c("norm", + "basic"), index = 2))

BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1024 bootstrap replicates

Note the use of the optional `index` argument to select each of the two parameters in the `boot` object created by `boot`.

Q68: Do the bootstrap estimates of the two parameters of the structural relation appear to be normally-distributed?  

Jump to A68 •
CALL:
boot.ci(boot.out = b.sr, conf = 0.95, type = c("norm", "basic"),
index = 2)

Intervals:
Level   Normal      Basic
95% (-1.635, -0.798) (-1.642, -0.788)

Calculations and Intervals on Original Scale

**Q69**: How well do the two types of confidence intervals agree?  Jump to **A69** •

**Challenge**: Use non-parametric bootstrapping to estimate the 95% confidence interval and best estimate of the correlation coefficient between grain and straw yields for this wheat variety grown in the conditions of Mercer & Hall’s experiment.

You should obtain the following plot of the bootstrap:

![Histogram of t](image)

You should obtain these results for the basic confidence interval and best estimate of the population correlation coefficient \( \rho \): \( \hat{\rho} = 0.7312, \rho \in (0.6818, 0.7817) \). Compare this to the parametric (Pearson’s) estimate: \( \hat{\rho} = 0.7298, \rho \in (0.686, 0.7683) \). What can you conclude about the appropriateness of the parametric test and its confidence interval computed on the basis of a theoretical bivariate normal distribution, in this experiment?

### 10.3 Answers

**A63**: The single (non-bootstrapped) estimate is 2.945, shown by the dashed vertical line, is close to the middle of the histogram. The histogram of the bootstrapped estimates of \( q_{0.01} \) is quite irregular. At low values there is a high frequency of some values and none of others; at high values a fairly uniform distribution. This
is because of the few low values in the sample.  

**A64**: The resampling only uses the known values; there are only 500 of these and so are not continuous.  

**A65**: The bootstrapped estimate is 2.938, while the one-sample estimate is 2.945. They are quite close, although the bootstrap estimate is a bit lower (i.e., more conservative, predicting a lower value for the 1%-lowest grain yields).  

**A66**: The normal approximation is 2.809...3.095; the basic bootstrap estimate is 2.84...3.08. In this case the basic estimate is a bit narrower than the normal approximation, probably because the very low values are not as likely as predicted by the normal distribution see the Q-Q normal probability plot of §7 where the lower tail is above the Q-Q line.  

**A67**: There is only a 5% chance that in the population of all possible small plots grown according to the Mercer & Hall experimental protocol, under the same conditions (weather, soil...), the lowest 1% of grain yields would be lower than 2.84 or higher than 3.08.  

**A68**: Yes, the parameters do seem normally-distributed. Thus the basic and normal confidence intervals should be almost the same.  

**A69**: The two types of confidence intervals agree very closely; this is because the bootstrap estimates are almost normally-distributed.  

We are done with these models, some variables, and the boot package, so clean up the workspace:

```r
> rm(model.grain.straw)
> rm(boot.q01, b.q01, b.q01.ci, boot.sr, b.sr, b.sr.ci)
> detach(package:boot)
```

11 Robust methods*

A **robust** inference is one that is not greatly disturbed by:

- a few unusual observations in the dataset, so-called **outliers**; or
- a “small” violation of model assumptions.

An example of the first case is a **contaminated** dataset, where some observations result from a **different process** then that which produced the others. In the present example this could be small areas of the wheat field with extreme stoniness, where a large amount of animal manure was stored, or where pests attacked. The issue here is that the observations do **not** all come
from the same population, i.e., result of a single process. But of course that
is not known prior to the experiment.

The second case (violation of model assumptions) can be the result of the
first case (outliers) but also because an incorrect model form was selected to
represent the underlying process.

Recall that in estimating linear regression parameters by least-squares, the
assumed model is:

\[ y_i = B X_i + \varepsilon_i \]

where the errors are identically and independently distributed (IID). If the
regression diagnostic plots suggest that this assumption is violated, robust
methods should be used.

11.1 A contaminated dataset

To illustrate the application of robust methods, we purposely add some
contamination to our dataset. That is, we simulate the case where the
wheat plants in some of the 500 plots were subject to some process other
than “normal” growth as affected by soil, weather, management, and small
attacks by a variety of pests. The contamination is that plots in one corner
of the field were attacked by deer, who ate most of the grain but did not
affect the straw\(^{16}\).

Q70: What could be some analysis options for the experimenter who
observes different processes in the field? Jump to A70

In the present example, we suppose that we receive the dataset without
having any opportunity to determine \textit{a priori} whether several processes were
active; we need to deal with the dataset as-is. Recall, the purpose of this
experiment is to investigate the distribution of many replications of grain
and straw yields when grown under identical conditions.

- Can we determine whether the conditions were “identical” except for
  identical random “noise”?
- How can we estimate true values for the “typical” situation when there
  is unknown contamination from another process?

Task 61: Make a “contaminated” version of the dataset by setting the
16 northwest-most plots’ grain yields (3.2%) to one-quarter of their actual
yields.

First, make a copy the “true” dataset:

\[ > \text{mhw.c <- mhw} \]

\(^{16}\)This happened to the one of author’s field experiments during his MSc research at the
Pennsylvania State University in the mid 1970’s.
Second, modify the NW corner. The dataset description (§A) states that the rows ran W to E, with 25 plots in each row, beginning at 1 on the W and running to 25 at the E, so that columns run S to N with 20 plots in each, running from 1 at the N to 20 at the S. We can find the 4 x 4 NW-most plots by selecting on row and column number:

```r
> ix <- (mhw.c$r < 5) & (mhw.c$c < 5)
> rownames(mhw.c[ix, ])
[1] "1" "2" "3" "4" "21" "22" "23" "24" "41" "42" "43" "44" "61"
[14] "62" "63" "64"
> subset(mhw.c, ix)
   r  c grain straw gsr in.north
1  1  1  3.63  6.37 0.56986 TRUE
2  2  1  4.07  6.24 0.65224 TRUE
3  3  1  4.51  7.05 0.63972 TRUE
4  4  1  3.90  6.91 0.56440 TRUE
21 1  2  4.15  6.85 0.60584 TRUE
22 2  2  4.21  7.29 0.57750 TRUE
23 3  2  4.29  7.71 0.55642 TRUE
24 4  2  4.64  8.23 0.56379 TRUE
41 1  3  4.06  7.19 0.56467 TRUE
42 2  3  4.15  7.41 0.56005 TRUE
43 3  3  4.40  7.35 0.59864 TRUE
44 4  3  4.05  7.89 0.51331 TRUE
61 1  4  5.13  7.99 0.64205 TRUE
62 2  4  4.64  7.80 0.59487 TRUE
63 3  4  4.69  7.50 0.62533 TRUE
64 4  4  4.04  6.66 0.60661 TRUE
```

The logical vector `ix` is a list of TRUE and FALSE, stating whether a given case (row, observation) in the dataframe is in the NW corner or not.

Now adjust the grain yields:

```r
> mhw.c[ix, "grain"] <- mhw.c[ix, "grain"]/4
```

**Task 62**: Summarize the effect of the contamination both numerically and graphically.

First the numeric summary, also the standard deviation:

```r
> summary(mhw$grain)

   Min. 1st Qu. Median Mean 3rd Qu. Max.
2.73  3.64  3.94  3.95  4.27  5.16

> sd(mhw$grain)

[1] 0.45828
```

> summary(mhw.c$grain)
Min. 1st Qu. Median Mean 3rd Qu. Max.
0.908 3.590 3.920 3.850 4.260 5.160

> sd(mhw.c$grain)
[1] 0.67636

Second, side-by-side histograms “before” and “after”:

> par(mfrow=c(1,2))
> hist(mhw$grain, xlab="Grain yield, lbs / plot",
+ main="Actual", breaks=seq(0,6, by=.25))
> rug(mhw$grain)
> hist(mhw.c$grain, xlab="Grain yield, lbs / plot",
+ main="Contaminated", breaks=seq(0,6, by=.25))
> rug(mhw.c$grain)
> par(mfrow=c(1,1))

11.2 Robust univariate modelling

Typical parameters estimated for a single variable are some measure of the center and spread. For a normally-distributed variable the appropriate measures are the mean and variance (or standard deviation). But these are not robust. The robust measures include the median for central tendency and the inter-quartile range (IQR) for spread.

Task 63: Compute the mean and standard deviation of the actual and contaminated grain yields. Also compute the robust measures. For all, compute the percent change due to contamination.

The mean, sd, median, and IQR functions compute these:

> mean(mhw$grain)
[1] 3.9486

> mean(mhw.c$grain)
3.8458
> (mean(mhw.c$grain) - mean(mhw$grain))/mean(mhw$grain) * + 100
[1] -2.6044
> sd(mhw$grain)
[1] 0.45828
> sd(mhw.c$grain)
[1] 0.67636
> (sd(mhw.c$grain) - sd(mhw$grain))/sd(mhw$grain) * 100
[1] 47.586
> median(mhw$grain)
[1] 3.94
> median(mhw.c$grain)
[1] 3.915
> (median(mhw.c$grain) - median(mhw$grain))/median(mhw$grain) * + 100
[1] -0.63452
> IQR(mhw$grain)
[1] 0.6325
> IQR(mhw.c$grain)
[1] 0.67
> (IQR(mhw.c$grain) - IQR(mhw$grain))/IQR(mhw$grain) * + 100
[1] 5.9289

Q71: How do the changes in the robust measures compare to those in the non-robust measures? Jump to A71 •

11.3 Robust bivariate modelling

In §8.1.1 we estimated the correlation between grain and straw yield. Recall that the parametric correlation (Pearson’s) should only be used if the two variables are distributed approximately bivariate normally. In the original dataset this seemed to be a reasonable assumption.
In case the relation can not be assumed to be bivariate normal, methods must be used that do not depend on this assumption. These are called non-parametric; for correlation a widely-used metric is Spearman’s $\rho$, which is the PPMC between the ranks of the observations.

**Task 64**: Compare the ranks of the grain and straw yields for the first eight plots in the original data frame.

The `rank` function returns the ranks of the values in a vector, from low to high. Ties can be handled in several ways; the default is to average the ranks. We display the data values and their ranks in a table:

```r
> head(cbind(mhw[, c("grain", "straw")], rank(mhw$grain), + rank(mhw$straw)), n = 8)
```

<table>
<thead>
<tr>
<th>grain</th>
<th>straw</th>
<th>rank(mhw$grain)</th>
<th>rank(mhw$straw)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.63</td>
<td>123.0</td>
<td>254.5</td>
</tr>
<tr>
<td>2</td>
<td>4.07</td>
<td>299.0</td>
<td>219.5</td>
</tr>
<tr>
<td>3</td>
<td>4.51</td>
<td>445.5</td>
<td>356.5</td>
</tr>
<tr>
<td>4</td>
<td>3.90</td>
<td>228.0</td>
<td>329.0</td>
</tr>
<tr>
<td>5</td>
<td>3.63</td>
<td>123.0</td>
<td>136.0</td>
</tr>
<tr>
<td>6</td>
<td>3.16</td>
<td>23.5</td>
<td>70.5</td>
</tr>
<tr>
<td>7</td>
<td>3.18</td>
<td>26.0</td>
<td>36.0</td>
</tr>
<tr>
<td>8</td>
<td>3.42</td>
<td>62.5</td>
<td>59.0</td>
</tr>
</tbody>
</table>

**Q72**: What are the data values and ranks of the first plot? Do the ranks of the two variables generally match?  

**Task 65**: Display a scatterplot of the ranks, alongside a scatterplot of the values of the original dataset, and below this, the same for the contaminated dataset.

```r
> par(mfrow = c(2, 2))
> plot(rank(mhw$grain), rank(mhw$straw), xlab = "Grain rank", + ylab = "Straw rank", pch = 1, main = "Original")
> plot(mhw$grain, mhw$straw, xlab = "Grain (lbs / plot)", + ylab = "Straw (lbs / plot)", pch = 20, main = "Original")
> plot(rank(mhw.c$grain), rank(mhw.c$straw), xlab = "Grain rank", + ylab = "Straw rank", pch = 1, main = "Contaminated")
> plot(mhw.c$grain, mhw.c$straw, xlab = "Grain (lbs / plot)", + ylab = "Straw (lbs / plot)", pch = 20, main = "Contaminated")
> par(mfrow = c(1, 1))
```
Q73: How similar are the rank and value scatterplots in both cases? Which scatterplot type (rank or value) is more affected by contamination? Jump to A73 •

Q74: Does the scatterplot of values (straw vs. grain yields) of the contaminated dataset appear to be bivariate normal? Jump to A74 •

Task 66: Compute the numerical value of the Spearman’s correlation between grain and straw yield, and compare it to the PPMC. Do this for both the original and contaminated datasets.

The cor and cor.test functions have an optional method argument; this defaults to pearson but can be set explicitly:

> (c.p <- cor(mhw$grain, mhw$c$straw, method = "pearson"))

[1] 0.72978

94
> (cc.p <- cor(mhw.c$grain, mhw.c$straw, method = "pearson"))
[1] 0.35968
> (c.s <- cor(mhw$grain, mhw.c$straw, method = "spearman"))
[1] 0.71962
> (cc.s <- cor(mhw.c$grain, mhw.c$straw, method = "spearman"))
[1] 0.61684

Q75: Which method is most affected by the contamination? Jump to A75

11.4 Robust regression

In §8.2.1 we computed the linear regression of straw yield as modelled by grain yield.

Task 67: Repeat the regression fit for the contaminated dataset, and compare it with the original estimates of the regression model parameters.

> print(model.straw.grain)
Call:
  lm(formula = straw ~ grain, data = mhw)
Coefficients:
  (Intercept) grain
    0.866  1.430

> (model.straw.grain.c <- lm(straw ~ grain, data = mhw.c))
Call:
  lm(formula = straw ~ grain, data = mhw.c)
Coefficients:
  (Intercept) grain
    4.678   0.478

Q76: How much did the regression parameters change? Jump to A76

Task 68: Display the regression diagnostic plots for the regression fit for the contaminated dataset. Compare these with the plots from §8.2.4.

> par(mfrow = c(2, 2))
> plot(model.straw.grain.c)
> par(mfrow = c(1, 1))
Q77: Are the assumptions for a least-squares linear fit justified? Which assumptions are violated? 

Task 69: Visualize the poor quality of the linear fit on a scatterplot of straw vs. grain yield of the contaminated dataset. Also show the linear fit to the original dataset.

We use the generic `plot` method with a formula to specify the scatterplot, the `abline` function to add lines extracted from the linear models, and the `legend` function to place a legend on the graph. Recall, the `with` function specifies an environment for the `plot` method, so that the field names can be written just as field names, not as dataframe and field.

```r
> with(mhw.c, plot(straw ~ grain))
> abline(model.straw.grain.c)
> abline(model.straw.grain, lty=2, col="blue")
> legend(1.5, 8.5, legend=c("fit", "fit to uncontaminated"),
> lty=1:2, col=c("black","blue"))
```
Q78: Looking at this scatterplot and the two lines, explain how the contamination affects the linear model fit.  

Many of the problems with parametric regression can be avoided by fitting a so-called “robust” regression line. There are many variants of this, well-explained by Birkes and Dodge [4] and illustrated with S code by Venables and Ripley [50]. Fox and Weisberg [18] is a good introduction to the concepts, illustrated by R code. 

Here we just explore one method: 1qs in the MASS package; this fits a regression to the “good” points in the dataset (as defined by some criterion), to produce a regression estimator with a high “breakdown” point. This method has several tuneable parameters; we will just use the default. 

This is the so-called “least trimmed squares” (LTS) estimate of the slope vector $\beta$, by the criterion of minimizing:

$$
\sum_{i=1}^{q} |y_i - x_i \beta|_{(i)}^2
$$

that is, the squared absolute deviations over some subset of the residuals, indicated by the subscript $(i)$ and the summand $q$ in the above formula. The smallest $q$ residuals are chosen as some proportion, by default:

$$
q = \lfloor (n + p + 1)/2 \rfloor
$$

where $p$ is the number of predictors and $n$ the number of observations (cases). In this case these are 2 and 500, so $q = 251$. There is no analytical solution, the method iterates: first fitting a regular linear model, then examining the residuals and selecting the $q$ smallest, then refitting.
(thus obtaining a new set of residuals), selecting the smallest $q$ of these, refitting, and so on until the fit converges.

---

**Task 70**: Load the MASS package and compute a robust regression of straw on grain yield. Compare the fitted lines and the coefficient of determination ($R^2$) of this with those from the least-squares fit.

```r
> require(MASS)
> (model.straw.grain.c.r <- lqs(straw ~ grain, data = mhw.c))
```

Call:
`lqs.formula(formula = straw ~ grain, data = mhw.c)`

Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>grain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.0465</td>
</tr>
<tr>
<td></td>
<td>1.6815</td>
</tr>
</tbody>
</table>

Scale estimates 0.548 0.531

```r
> sqrt(mean(residuals(model.straw.grain.c.r)^2))
```

[1] 0.61354

The scale estimates are the scale of the error, similar to the residual mean square in a least-squares fit (shown above for comparison). There are two scale estimates, the first is based on the fit and is more conservative (larger).

---

**Task 71**: Visualize the robust fit on a scatterplot of straw vs. grain yield of the contaminated dataset. Also show the linear fit to the contaminated and original datasets.

```r
> with(mhw.c, plot(straw ~ grain))
> abline(model.straw.grain.c.r)
> abline(model.straw.grain.c, lty = 3, col = "red")
> abline(model.straw.grain, lty = 2, col = "blue")
> legend(1.5, 8.5, legend = c("robust fit", "linear fit",
+ "fit to uncontaminated"), lty = c(1, 3, 2), col = c("black",
+ "red", "blue"))
```
Q79: Describe the effect of the robust fit.  

Challenge: Compute the robust regression for the original (uncontaminated) dataset. How much does the fit change? What does this imply about the “outliers” identified in §4.2? Are they contamination (i.e., from another process) or just unusual (extreme) observations?

Challenge: Repeat the analysis of this section with a larger contaminated portion of the observations. At what point do the robust estimates also become unreliable?

11.5 Answers

A70: Some possibilities:

(1) Remove the known contamination and analyze the “typical” case; state that the results only apply to these.

(2) Mark observations as “typical” or “contaminated” and use this factor in modelling.

A71: Since the contaminated observations are smaller than the original, the central tendency will be lower. The contaminated mean is lower by -2.6% whereas the contaminated median is hardly affected, only -0.6% lower.

The measure of spread is much more affected by contamination: the standard deviation increases dramatically, by 48%, whereas the IQR only increases by 6%.
A72 : The first plot has grain and straw yields of 3.63 and 6.37, respectively; these are ranked 123 and 254.5 in their respective vectors. The grain yield is thus just under the first quartile (123 out of 500) whereas the straw yield is higher, just under the median (254 out of 500). For this plot, the straw yield ranks considerably higher than the grain yield. Overall, in the eight displayed ranks, there is a clear correspondence, but not very close. Return to Q72

A73 : In both datasets the scatterplot of ranks is more diffuse; this is because the ranks throw away much information (the actual numeric values). In the case of bivariate normal distributions, the tails (extreme values) contribute disproportionately to the correlation. The rank plots are much less affected by contamination than the value plots; thus we expect a non-parametric correlation (based on ranks) to be less affected. Return to Q73

A74 : The relation of straw and grain yields is definitely not bivariate normal. The group of observations at low grain yields (around 1 lb plot\(^{-1}\)) has very little relation with straw yield, and also is far from the scatter of the “normal” plots. Return to Q74

A75 : The parametric estimate of correlation (Pearson’s) is greatly affected, by -51%; whereas the non-parametric estimate (Spearman’s) is only changed by -14%. Return to Q75

A76 : The regression parameters (slope and intercept) both change drastically. The slope decreased from 1.43 to 0.48, i.e., from well above unity (straw yield increases more than grain yield) to well below (the reverse). The intercept then is adjusted to maintain the centroid of the regression; for the original dataset this is 0.87 lb plot\(^{-1}\) straw, i.e., a grain yield of zero corresponds to a straw yield of a bit under 1 lb plot\(^{-1}\) straw, but to compensate for the greatly reduced slope in the model of the contaminated dataset this increases to 4.68 lb plot\(^{-1}\) straw. Return to Q76

A77 : Several assumptions are clearly violated:

(1) The residuals have a clear relation with the fits: low fits have very high positive residuals (these are the plots where the deer ate most of the grain); to compensate there is then a linear relation between residuals and fits from about 6 to 7.5 lbs plot\(^{-1}\) straw, i.e., the unaffected plots.

(2) The distribution of the residuals is not normal; this is especially clear at high residuals where the actual residuals are too high, i.e., large over-predictions (because the model is fitting mostly the ordinary relation, the low-grain plots are over-predicted).

(3) The high residuals are also associated with high leverage, i.e., the low-grain plots disproportionately affect the regression parameters (which is why the slope is
A78: This scatterplot clearly shows what we mean by “leverage”: the 16 contaminated points “pull” the original line (shown as a dashed line on the plot) towards them. The further from the original centroid, the more the leverage, exactly as with a physical lever.

A79: The robust fit models the uncontaminated portion of the dataset and completely ignores the contaminated observations. Thus it gives a true model of the dominant process of wheat growth. The robust fit is a bit different from the fit to the uncontaminated dataset, because it also ignores the outliers in the original dataset.

12 Multivariate modelling

In §8.2 we determined that the regression of straw yield on grain yields, over the whole field, was significant: i.e. straw yield does vary with grain yield. In §9 we determined that straw yields of the two field halves differ significantly. This raises the question whether it is possible and useful to combine these two findings, i.e. whether straw yield can be modelled as a function of both field half and grain yield.

There are four possibilities, from simplest to most complex:

1. Straw yields can be modelled by field half only;
2. Straw yields can be modelled by grain yield only;
3. Straw yields can be modelled by field half and grain yield, additively (i.e. with these as independent predictors);
4. Straw yields must be modelled by field half and grain yield, considering also the interaction between them (i.e. the relation between straw and grain yield differs in the two field halves).

These are tested with increasingly-complex linear models:

1. One-way ANOVA (or, two-sample t-test) of straw yield; following the evidence in the boxplot of §9.1 that seemed to show slightly higher yields in the north half; this was computed in §9.3 above;
2. Linear model of straw yield predicted by grain yield only, following the evidence of the scatterplot, ignoring field halves; this was computed in §8.2 above;
3. Linear model of straw yield predicted by field half (a categorical variable) and grain yield (a continuous variable), with an additive model (§12.1, below);
4. Linear model of straw yield predicted by field half (a categorical variable) and grain yield (a continuous variable), with an interaction model (§12.3, below);
We now compute all of these and compare the proportion of the total variation in straw yield that each explains.

**Task 72**: Display the results of the field-half and bivariate regression models.

Straw yield vs. field half was already computed in §9.3 above and saved as model object `model.straw.ns`:

```r
> summary(model.straw.ns)
Call:
  lm(formula = straw ~ in.north, data = mhw)
Residuals:
   Min     1Q Median     3Q    Max
-2.181  -0.608  -0.108   0.572   2.359
Coefficients:
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)  6.7484     0.0549   122.90 < 2e-16 ***
in.northTRUE -0.4671     0.0777    -6.02   3.5e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.868 on 498 degrees of freedom
Multiple R-squared: 0.0677, Adjusted R-squared: 0.0659
F-statistic: 36.2 on 1 and 498 DF,  p-value: 3.48e-09
```

**Q80**: Is there evidence that average straw yield differs in the two field halves? What proportion of the total variation in straw yield is explained by field half? Jump to A80

Straw yield vs. grain yield was computed in §8.2 above and saved as model object `model.straw.grain`:

```r
> summary(model.straw.grain)
Call:
  lm(formula = straw ~ grain, data = mhw)
Residuals:
   Min     1Q Median     3Q    Max
 -2.0223  -0.3529  0.0104  0.3734  3.0342
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.8663     0.2387    3.63  0.00031 ***
grain        1.4305     0.0601   23.82 < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.615 on 498 degrees of freedom
```

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Q81: Is there evidence that straw yield varies with grain yield? What proportion of the total variation in straw yield is explained by grain yield?

Since both of these predictors (field half and grain yield) do explain some of the straw yield, it seems logical that a combination of the two, i.e. a multivariate model, might explain more than either separately. So, we now model straw yield vs. grain yield, also accounting for the overall difference between field halves.

12.1 Additive model: parallel regression

The simplest multivariate model is an additive model, also called parallel regression because it fits one regression line, but with the intercept at different levels, one for each field half.

Task 73: Model straw yield as the combined effect of two independent predictors: field half and grain yield.

We use the `lm` function, naming both predictors on the right-hand side of the model formula, combined with the `+` “additive effects” formula operator. This is not an arithmetic `+` (addition), because it is written in a model formula.

```r
> model.straw.ns.grain <- lm(straw ~ in.north + grain,
+   data = mhw)
> summary(model.straw.ns.grain)
```

Call:
`lm(formula = straw ~ in.north + grain, data = mhw)`

Residuals:
```
     Min      1Q  Median      3Q     Max
-2.2548 -0.3189 -0.0276  0.3042  2.7871
```

Coefficients:
```
             Estimate Std. Error t value Pr(>|t|)  
(Intercept)  1.0461     0.2178     4.8 2.1e-06 ***
in.northTRUE -0.5132     0.0500    -10.3 < 2e-16 ***
grain        1.4499     0.0546     26.5 < 2e-16 ***
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Residual standard error: 0.559 on 497 degrees of freedom
Multiple R-squared: 0.559, Adjusted R-squared: 0.557
F-statistic: 396 on 2 and 497 DF, p-value: <2e-16
Q82: Is there evidence that both field half and grain yield are needed to predict straw yield? What proportion of the total variation in straw yield is explained by this model?

Q83: According to the model summary, what is the difference in overall yields between the S and N halves? What is the slope of the regression line for straw vs. grain yield? Is this the same as the slope for the model that does not include field half?

In parallel regression (additive effects of a continuous and discrete predictor) there is only one regression line, which is displaced up or down for each class of the discrete predictor. Even though there are two predictors, we can visualize this in a 2D plot by showing the displaced lines.

```r
> with(mhw, plot(straw ~ grain, col = ifelse(in.north,
+   "blue", "slategray"), pch = 20, xlab = "grain (lbs plot-1)",
+   ylab = "straw (lbs plot-1")
> title(main = "Straw vs. grain yield")
> title(sub = "N half: blue, S half: grey; whole-field line: red")
> abline(coefficients(model.straw.ns.grain)["(Intercept)"],
+   coefficients(model.straw.ns.grain)["grain"], col = "slategray")
> abline(coefficients(model.straw.ns.grain)["(Intercept)"] +
+   coefficients(model.straw.ns.grain)["in.northTRUE"],
+   coefficients(model.straw.ns.grain)["grain"], col = "blue")
> abline(model.straw.grain, lty = 2, col = "red")
> grid()
```

In the code for this plot, note the use of the `coefficients` function to extract the model coefficients.
12.2 Comparing models

Is a more complex model better than a simpler one? There are several ways to answer this, among which are:

- Compare the adjusted $R^2$ of the two models: this is the proportion of the variance explained;
- Directly compare hierarchical models with an Analysis of Variance.

---

**Task 74**: Compare the adjusted $R^2$ of the three models.

We’ve already seen these in the model summaries; they can be accessed directly as field `adj.r.squared` of the model summary:

```r
> summary(model.straw.ns)$adj.r.squared
[1] 0.065864
> summary(model.straw.grain)$adj.r.squared
[1] 0.53164
> summary(model.straw.ns.grain)$adj.r.squared
[1] 0.61268
```

---

**Q84**: Which model explains the most variability? [Jump to A84]

Another way to compare two hierarchical models (i.e. where the more complex model has all the predictors of the simpler one) is with an analysis of variance: comparing variance explained vs. degrees of freedom. This is a statistical test, so we can determine whether the more complex model is provably better.

---

**Task 75**: Compare the additive multivariate model to the two univariate models.

The `anova` function can be used to compare the models:

```r
> anova(model.straw.ns.grain, model.straw.ns)
```

**Analysis of Variance Table**

```
Model 1: straw ~ in.north + grain
Model 2: straw ~ in.north
          Res.Df RSS Df Sum of Sq F Pr(>F)
1       497 155
2     498  375  -1    -220 704 <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
> anova(model.straw.ns.grain, model.straw.grain)
```

---

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Analysis of Variance Table

Model 1: straw ~ in.north + grain
Model 2: straw ~ grain

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>497</td>
<td>1</td>
<td>155</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>498</td>
<td>-1</td>
<td>-32.9</td>
<td>105</td>
<td>&lt;2e-16 ***</td>
</tr>
</tbody>
</table>

Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

The `anova` function compares the residual sum of squares (RSS) of the two models; this is the amount of variability not explained by the model, so a lower RSS is better. It then computed the F-ratio between the two variances, and the probability that this large an F-value, with the degrees of freedom (d.f.) could occur by chance, if the null hypothesis of no model improvement is true. The probability of a Type I error (falsely rejecting a true null hypothesis) is reported as field `Pr(>F)`; lower is better.

Q85: Is the multivariate additive model provably better than either univariate model? Jump to A85

12.3 Interaction model

We saw that the additive model is superior to either single-predictor model. However, there is also the possibility that both field half and grain yield help predict straw yield, but that the relation between straw and grain is different in the two halves; this is known as an interaction. This allows a different linear regression in each field half, rather than a parallel regression.

Q86: What is the difference between an additive and interaction model, with respect to processes in the field? Jump to A86

Task 76: Model straw yield as the combined effect of two interacting predictors: field half and grain yield.

We use the `lm` function, naming both predictors on the right-hand side of the model formula, combined with the `*` “interactive effects” formula operator. This is not an arithmetic `*` (multiplication).

```r
> model.straw.ns.grain.i <- lm(straw ~ in.north * grain, + data = mhw)
> summary(model.straw.ns.grain.i)
```

Call:
`lm(formula = straw ~ in.north * grain, data = mhw)`

Residuals:
```
        Min       1Q  Median       3Q      Max
```
Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| (Intercept) | 1.2930     | 0.2979  | 4.34     | 1.7e-05 *** |
| in.northTRUE | -1.0375    | 0.4350  | -2.39    | 0.017 *     |
| grain      | 1.3872     | 0.0752  | 18.44    | <2e-16 ***  |
| in.northTRUE:grain | 0.1328 | 0.1094 | 1.21 | 0.225 |

---

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.559 on 496 degrees of freedom
Multiple R-squared: 0.615, Adjusted R-squared: 0.613
F-statistic: 265 on 3 and 496 DF, p-value: <2e-16

Q87: What do the four coefficients in the regression equation represent?

Jump to A87

Q88: Are the two slopes (one for each field half) significantly different? Is an interaction model indicated? What does that imply about the processes in the field?

Jump to A88

Even though the slopes are not significantly different, we show them graphically, to visualize how different they are.

Task 77: Plot the two regressions (one for each field half).

To do this, we use the optional subset argument to the lm method to select just some observations, in this case, those in a zone. We plot each regression line (using the abline function on the model object returned by lm) and its associated points in different colours, using the col graphics argument. Dashed lines (using the lty graphics argument) show the parallel regression for the two field halves.

```r
> with(mhw, plot(straw ~ grain, col = ifelse(in.north, "blue", "slategray"), pch = 20, xlab = "grain (lbs plot-1)", ylab = "straw (lbs plot-1)"))
> title(main = "Straw vs. grain, by field half")
> title(sub = "Interaction: solid lines; additive: dashed lines")
> abline(lm(straw ~ grain, data = mhw, subset = in.north), col = "blue")
> abline(lm(straw ~ grain, data = mhw, subset = !in.north), col = "slategray")
> abline(coefficients(model.straw.ns.grain)["(Intercept)"], + coefficients(model.straw.ns.grain)["grain"], col = "slategray", + lty = 2)
> abline(coefficients(model.straw.ns.grain)["(Intercept)"] + + coefficients(model.straw.ns.grain)["in.northTRUE"], + coefficients(model.straw.ns.grain)["grain"], col = "blue", + lty = 2)
```
Is this more complex interaction model significantly better than the additive model?

**Task 78**: Compare the interaction and additive models by their adjusted $R^2$ and with an analysis of variance.

```r
> summary(model.straw.ns.grain)$adj.r.squared
[1] 0.61268

> summary(model.straw.ns.grain.i)$adj.r.squared
[1] 0.61305

> anova(model.straw.ns.grain.i, model.straw.ns.grain)

Analysis of Variance Table

Model 1: straw ~ in.north * grain
Model 2: straw ~ in.north + grain

Res.Df RSS Df Sum of Sq F Pr(>F)
1 496 155
2 497 155 1 -0.46 1.47 0.23
```

**Q89**: Does the more complex model have a higher proportion of variance explained? Is this statistically significant?  

Jump to A89 •

### 12.4 Regression diagnostics

As with univariate regression, multivariate regression models must be examined to see if they meet modelling assumptions.
**Task 79**: Display the diagnostic plots for the additive model: (1) residuals vs. fits; (2) normal Q-Q plot of the residuals; (3) residuals vs. leverage; (4) Cook’s distance vs. leverage.

These are plot types 1, 2, 5, and 6, respectively, selected with the `which` optional argument to the `plot` function applied to linear model output. We also specify the number of extreme points to label with the `id.n` optional argument.

```r
> par(mfrow = c(2, 2))
> plot(model.straw.ns.grain, which = c(1, 2, 5, 6), id.n = 10)
> par(mfrow = c(1, 1))
```

**Q90**: Which observations (plots) are marked on the plots as being potential problems?  

*Jump to A90*

We can identify these numerically. First, we examine plots that were not well-modelled. We use the criterion of model residuals more than three standard deviations from zero; we want to see if there is any pattern to these.
We identify the plots with the large residuals, using the `rstandard` “standardized residuals” function, and show just these records in the data frame, using the `which` function to identify their row (record) numbers:

```r
> (selected <- which(abs(rstandard(model.straw.ns.grain)) > + 3))
15 35 184 285 292 295 311 337 362
15 35 184 285 292 295 311 337 362
> rstandard(model.straw.ns.grain)[selected]
 15 35 184 285 292 295 311 337
 362
 3.4074
```

Second, build a data frame with all the information for these plots, along with the residuals, using the `cbind` function to add a column:

```r
> mhw.hires <- cbind(mhw[selected,], + sres = rstandard(model.straw.ns.grain)[selected])
> rm(selected)
> str(mhw.hires)
'data.frame': 9 obs. of 7 variables:
$ r : int 15 15 4 5 12 15 11 17 2
$ c : int 1 2 10 15 15 15 16 17 19
$ grain : num 3.46 4.42 4.59 3.7 4.86 3.73 3.74 3.05 4.26
$ straw : num 8.85 5.2 5.41 7.67 6.39 8.58 8.63 7.64 8.61
$ gsr : num 0.391 0.85 0.848 0.482 0.761 ...
$ in.north: logi FALSE FALSE TRUE TRUE FALSE FALSE ...
$ sres : num 5 -4.05 -3.19 3.18 -3.06 ...
```

Finally, order the selected plots by the residual, using the `order` function:

```r
> mhw.hires[order(mhw.hires$sres),]
   r c grain straw gsr in.north sres
 35 15 2 4.42 5.20 0.85000 FALSE -4.0459
184 4 10 4.59 5.41 0.84843 TRUE -3.1930
292 12 15 4.86 6.39 0.76056 FALSE -3.0646
285 5 15 3.70 7.67 0.48240 TRUE 3.1776
362 2 19 4.26 8.61 0.49477 TRUE 3.4074
295 15 15 3.73 8.58 0.43473 FALSE 3.8105
311 11 16 3.74 8.63 0.43337 FALSE 3.8741
337 17 17 3.05 7.64 0.39921 FALSE 3.9068
15 15 1 3.46 8.85 0.39096 FALSE 5.0007
```

We can also visualize the locations of these in the field: high positive residuals green, high negative residuals red, symbol size proportional to the grain/straw ratio:

```r
> with(mhw, plot(c, r, ylim = c(20, 1), cex = 3 * gsr/max(gsr), + pch = 20, col = ifelse(rstandard(model.straw.ns.grain) > + 3, "brown", ifelse(rstandard(model.straw.ns.grain) < + (-3), "red", ifelse(in.north, "lightblue", "gray"))),
```

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The plot of leverage vs. Cook’s distance shows which plots most affect the fit: high leverage and high distance means that removing that plot would have a large effect on the fit.

Q91: Are the high positive or negative residuals concentrated in one part of the field? Is there anything else unusual about these? Hint: look at the most negative and positive residuals.

The plot of leverage vs. Cook’s distance shows which plots most affect the fit: high leverage and high distance means that removing that plot would have a large effect on the fit.

Q92: Do the two highest-residual plots identified in the previous question have high leverage? Which high-residual plots also have high leverage?

We can examine the effect of these on the fit by re-fitting the model, leaving out one or more of the suspect plots.

Task 80: Fit the model without the two adjacent plots where we hypothesize sloppy field procedures and compare the goodness-of-fit and regression equations to the original model.

To exclude some observations, we use the - operator on a list of dataframe row numbers created with the c function:
> model.straw.ns.grain.adj <- lm(straw ~ in.north + grain,
+     data = mhw[-c(15, 35), ])
> summary(model.straw.ns.grain.adj)

Call:  
  lm(formula = straw ~ in.north + grain, data = mhw[-c(15, 35), ])

Residuals:
   Min     1Q   Median     3Q    Max 
-1.7926 -0.3106  -0.0274  0.3017  2.1942 

Coefficients:
                       Estimate Std. Error  t value  Pr(>|t|) 
(Intercept)            0.9528     0.2094    4.55 6.8e-06 ***
in.northTRUE          -0.5118     0.0481   -10.64  < 2e-16 ***
grain                  1.4731     0.0525    28.04  < 2e-16 ***

---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.536 on 495 degrees of freedom 
Multiple R-squared: 0.64, Adjusted R-squared: 0.638 
F-statistic: 440 on 2 and 495 DF,  p-value: <2e-16 

> summary(model.straw.ns.grain)

Call: 
  lm(formula = straw ~ in.north + grain, data = mhw)

Residuals:
   Min     1Q   Median     3Q    Max 
-2.2548 -0.3189  -0.0276  0.3042  2.7871 

Coefficients:
                       Estimate Std. Error  t value  Pr(>|t|) 
(Intercept)            1.0461     0.2178    4.8  2.1e-06 ***
in.northTRUE          -0.5132     0.0500   -10.3  < 2e-16 ***
grain                  1.4499     0.0546    26.5  < 2e-16 ***

---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.559 on 497 degrees of freedom 
Multiple R-squared: 0.614, Adjusted R-squared: 0.613 
F-statistic: 396 on 2 and 497 DF,  p-value: <2e-16 

Q93: Does the model without the two plots fit the remaining plots better than the original model? How different are the model coefficients?  Jump to A93  

Challenge: Compare the four diagnostic plots for the adjusted additive regression model (i.e., leaving out the “suspect” points) with the diagnostic plots for the additive regression model with all points, above (§12.4). Display the two sets of diagnostic plots together and evaluate them visually. What,
if anything, has improved? Does the model now meet the assumptions of linear regression?

12.5 Analysis of covariance: a nested model*

In the parallel-lines model there is only one regression line between the continuous predictor and predictand, which can be moved up and down according to different class means; this is an additive model. In the interaction model there is both an overall line and deviations from it according to class, allowing different slopes, as well as differences in class means.

Another way to look at this is to abandon the idea of a single regression altogether, and fit a separate line for each class. This is a nested model: the continuous predictor is measured only within each level of the classified predictor. There is no interest in the whole-field relation between straw and grain, only the overall difference between classes (here, the field halves), and then the best fit of the straw vs. grain relation in each half separately.

A nested model is specified with the / formula operator (this is not mathematical division). This is to be read as “fit the relation after the / separately for the two values of the classified variable”.

```r
> model.straw.ns.grain.nest <- lm(straw ~ in.north/grain, 
+   data = mhw)
> summary(model.straw.ns.grain.nest)

Call:
  lm(formula = straw ~ in.north/grain, data = mhw)

Residuals:
   Min     1Q   Median     3Q    Max
-2.2242 -0.3169  -0.0398  0.3136  2.7574

Coefficients:
             Estimate Std. Error   t value Pr(>|t|)
(Intercept)  1.2930      0.2979    4.34  1.7e-05 ***
in.northTRUE -1.0375      0.4350   -2.39   0.017 *
in.northFALSE:grain 1.3872      0.0752   18.44  < 2e-16 ***
in.northTRUE:grain 1.5199      0.0794   19.14  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.559 on 496 degrees of freedom
Multiple R-squared: 0.615,   Adjusted R-squared: 0.613
F-statistic: 265 on 3 and 496 DF,  p-value: <2e-16

> plot(straw ~ grain, data = mhw, col = ifelse(mhw$in.north, 
+   "blue", "slategray"), pch = 20, xlim = c(2.5, 5.5),
+   ylim = c(4, 9.5))
> coef <- coef(model.straw.ns.grain.nest)
> abline(coef[1], coef[3], col = "slategray")
> abline(coef[1] + coef[2], coef[4], col = "blue")
```
The nested model does not compute an overall slope of straw vs. grain; instead each half has its own regression line (intercept and slope). The coefficient `in.northTRUE` gives the difference between the intercept of the N-half regression line from the intercept of the S-half regression line. The two coefficients `in.northFALSE:grain` and `in.northTRUE:grain` give the computed slopes of the two regression lines of straw yield vs. grain yield.

Q94: What are the two slopes of straw vs. grain? Are they different? Do they differ from the single slope found in the parallel regression model?  

Q95: This model has the same adjusted R-squared as the interaction model. Why?

Challenge: Compare the regression lines of this nested model with the regression lines implied by the interaction model. Are they the same? Why or why not?

12.6 Answers

A80: Yes, straw yield most likely \((p \approx 0)\) is higher in the South half; but this explains very little (0.066) of the total variance.  

A81: Yes, straw yield almost surely \((p \approx 0)\) varies with grain yield; this explains about half (0.532) of the total variance.

A82: Both coefficients are highly significant (probability that they are really
zero almost nil). Coefficient in.northTRUE represents the difference between field halves, and grain the regression between straw and grain.

This explains over 60% (0.613) of the total variance.  

A83: Coefficient in.northTRUE represents the difference between field halves; the fitted value is -0.5132 lb. plot^{-1}. Coefficient grain is the regression between straw and grain; the fitted value is 1.4499 lb. straw increase per plot for each lb grain increase per plot. This is not the same as the best-fit univariate line (ignoring field half), which is 1.4305.  

A84: The multivariate additive model is clearly best; it explains about two-thirds (66%) of the variability, whereas the whole-field straw vs. grain model only explains just more than half (53%) and the field-half model very little (6%).  

A85: Yes, in both cases the probability that we’d be wrong by rejecting the null hypothesis of no difference is practically zero. The RSS decreases from 375.4 for the field-half model, and 188.2 for the whole-field straw vs. grain model, to 155.3 for the combined model. That is, much less variability in straw yield remains unexplained after the combined model.  

A86: The parallel regression models the case where one half of the field is more productive, on average, than the other, but the relation between grain and straw is the same in both cases (i.e. the grain/straw ratio is the same in both field halves). There is no difference in plant morphology, just overall size.

By contrast, the interaction model allows that the two field halves may also differ in the grain/straw ratio, i.e. the relation between grain and straw yield – different plant morphology.  

A87: The coefficients are:

1. (Intercept): estimated straw yield at zero grain yield and in the S field half;  
2. in.northTRUE: difference in average yield in the N vs. the S;  
3. grain: increase in straw yield for each unit increase in grain yield, in the S field half;  
4. in.northTRUE:grain: difference in slope (increase in straw yield for each unit increase in grain yield) in the N vs. the S.

A88: The coefficient in.northTRUE:grain is not significant; the probability of falsely rejecting the null hypothesis is quite high, 0.2255, so we should accept the hypothesis that this difference is really 0.
According to this test, the interaction is not significant. Plants grow with the same morphology in the two field halves. 

A89: The interaction model has slightly higher adjusted $R^2$: 0.61305 vs. 0.61268. However, the ANOVA table shows that this increase has a $p=0.225$ probability of occurring by chance, so we conclude the interaction model is not justified. This is the same conclusion we reached from the model summary, where the interaction term (coefficient) was not significant. 

A90: Plots labelled on the diagnostic graphs are 15, 337, 311, 295, 362, 285 (positive residuals) and 35, 184, 292, 457 (negative residuals). These have residuals more extreme than expected by theory (normal Q-Q plot of residuals). Plots 292, 337, 264, 184 and 309 are large residuals with high leverage. 

A91: The highest four positive residuals are all in the S half, but otherwise do not seem clustered. The most negative residual is from plot 35 ($r=15, c=2$) and the most positive from the immediately adjacent plot 15 ($r=15, c=1$). Could some of the grain from plot 15 have been accidentally measured as part of the yield of plot 35? If these two are combined, the grain/straw ratio is 0.56085, close to the mean grain/straw ratio of the whole field, 0.61054. 

A92: Plots 15 and 35 do not have high leverage, i.e. their removal would not change the equation very much. The high-leverage plots that also have high residuals are 292 and 184 (negative residuals) and 337, 264 and 309 (positive residuals). 

A93: The fit is considerably better without these badly-modelled plots: 0.63841 without the two plots vs. 0.61268 with them. Another 2.5% of the variation is explained.

The coefficient for field half hardly changes, but the regression line changes substantially: higher intercept: (0.9528 vs. 1.0461) and shallower slope (1.4731 vs. 1.4499). As predicted by the high leverage, removing these points changes the functional relation. 

A94: The slope in the S half is 1.3872, in the N half 1.5199. These differ considerably from each other, and from the parallel regression slope: 1.4499. The slope in the S half is less steep, in the N half steeper, than the parallel regression slope. 

A95: Both models have four parameters to fit the same dataset. Both model difference between levels (either means or intercepts) and slopes. We are done with these models and some other variables, so clean up the workspace:

```r
> rm(model.straw.ns, model.straw.grain, model.straw.ns.grain, 
+ model.straw.ns.grain.adj, model.straw.ns.grain.i,
```
13 Principal Components Analysis

In §8.1.1 and §11.3 we computed the correlation between the grain and straw yields from each plot, i.e., the strength of their association. This showed that they are highly-correlated. Thus there is redundancy: the two variables are not fully independent.

Principal components analysis (PCA) is a data reduction technique. It finds a new set of variables, equal in number to the original set, where these so-called synthetic variables are uncorrelated. In addition, the first synthetic variable represents as much of the common variation of the original variables as possible, the second variable represents as much of the residual variation as possible, and so forth. This technique thus reveals the structure of the data. The transformation itself and the synthetic variables produced by it can be interpreted by the analyst to understand the underlying processes.

In the present example, Mercer & Hall measured two variables: grain and straw yield. However, these measured quantities are the outcomes of processes which we can not directly observe: (1) plant growth; (2) partition of plant growth between grain and straw. PCA can be used to gain insight into these.

PCA is often used for data reduction in datasets with many variables; good examples are image processing or spectroscopy with many bands, and geochemical datasets with many measured elements. It is explained by many textbooks on remote sensing [e.g. 1, 33]

Note: In terms of mathematics, the vector space made up of the original variables is projected onto another space such that the projected variables are orthogonal, with descending variances. The mathematics are well-explained in many texts, e.g., [8, 29].

PCA can be standardized or not. If so, the variables are centred to zero mean and then scaled to unit variance, thus giving them equal importance mathematically. In the second, the original units are used after centring; thus variables with more variance are more important. In the present case, although the two variables are computed in the same units of measure (lb. plot^{-1}), they are of intrinsically different magnitude (there is much more straw than grain). To reveal the relation between the variables they should be standardised before computing the principal components.

Task 81: Compute the standardized PCA of grain and straw yields.

The `prcomp` function computes the PCA; with the optional `scale` argument the variables are first scaled, resulting in standardized PCA. The output is an object of class `prcomp`.

```r
> pc <- prcomp(mhw[, c("grain", "straw")], scale = T)
> summary(pc)
```

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Importance of components:

<table>
<thead>
<tr>
<th></th>
<th>PC1</th>
<th>PC2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>1.315</td>
<td>0.520</td>
</tr>
<tr>
<td>Proportion of Variance</td>
<td>0.865</td>
<td>0.135</td>
</tr>
<tr>
<td>Cumulative Proportion</td>
<td>0.865</td>
<td>1.000</td>
</tr>
</tbody>
</table>

The standard deviations shown here are of the synthetic variables (principal components); these re-apportion the standard deviations of the original variables according to the redundancy.

Q96: How much of the variance is explained by each component? How much data redundancy is in the original two variables? Jump to A96

Sometimes PCA is used just for data reduction, but here our interest is in the interpretation. One way to interpret is to examine the loadings: these show how the synthetic variables are created from the original ones.

Note: Mathematically, these are the eigenvectors (in the columns) which multiply the original (scaled and centred) variables to produce the synthetic variables (principal components).

Task 82: Examine the loadings of the two PC’s.

These are in the rotation field of the PCA object.

> pc$rotation

<table>
<thead>
<tr>
<th></th>
<th>PC1</th>
<th>PC2</th>
</tr>
</thead>
<tbody>
<tr>
<td>grain</td>
<td>0.70711</td>
<td>-0.70711</td>
</tr>
<tr>
<td>straw</td>
<td>0.70711</td>
<td>0.70711</td>
</tr>
</tbody>
</table>

These can be visualized on a so-called biplot, which shows the loadings as vectors in the space spanned by two PC’s. It also shows the location of the 500 observations in this space, i.e., the values of the synthetic variables (called the principal component “scores”).

Task 83: Display the biplot.

This is displayed with the biplot method, which when presented with an object of class prcomp specializes to the biplot.prcomp function. We specify the pc.biplot argument to be TRUE, to produce the biplot as proposed by Gabriel [19], which scales the scores of the observations and rotations by the components’ standard deviations (eigenvalues).

> biplot(pc, pc.biplot = T)
> abline(h = 0, lty = 2)
> abline(v = 0, lty = 2)
> grid()
The biplot can be interpreted in several ways:

1. The **orientation** (direction) of the vector, with respect to the PC space. The more a vector, which represents an original variable, is parallel to a PC axis, the more it contributes to that PC.

2. The **length** in the space defined by the displayed PCs; the longer the vector, the more variability of this variable is represented by the two displayed PCs.

3. The **angles between vectors** of different variables show their correlation in the space spanned by the two displayed PC’s: small angles represent high positive correlation, right angles represent lack of correlation, opposite angles represent high negative correlation.

The scaled values of the eigenvectors are shown as the top (PC1) and right (PC2) axes.

Because of the scaling applied, the inner (vector) product between two variables approximates their covariance (if standardized, their correlation). Thus two vectors that are almost parallel will have a very high covariance, and two orthogonal vectors will have no covariance.

4. The **location of observations** in the biplot space, defined by their scores scaled by the standard deviation of each PC, i.e., its eigenvalue: this shows the relation of observations to each other and which observations are unusual. These values are shown on the bottom (PC1) and left (PC2) axes.

Because of the scaling applied, the distances between observations plot-
ted in this space approximate the Mahalanobis distance between them; this is a common multivariate measure of similarity between observations.

Q97: Considering the biplot and loadings matrix, what does the first PC represent? Jump to A97

To help answer this question, we can examine the field plots with the highest and lowest scores of PC1. The scores (synthetic variables) are returned in the x field of the `prcomp` object; this is a matrix whose columns are the PC scores.

The `which.max` and `which.min` functions find the positions of the maximum and minimum values in a vector. We then extract the record from the dataframe, and also show the PC scores:

```r
> summary(pc$x)

    PC1     PC2
Min. : -3.6114  Min. : -1.8592
1st Qu.: -0.9321 1st Qu.: -0.3474
Median :  0.0996  Median :  0.0119
Mean :   0.0000  Mean :   0.0000
3rd Qu.:  0.9220 3rd Qu.:  0.3028
Max. :   3.6521  Max. :   2.5921

> mhw[ix.max <- which.max(pc$x[, "PC1"], ]
   r c grain straw gsr in.north
79 19 45.16 8.78 0.5877 FALSE

> mhw[ix.min <- which.min(pc$x[, "PC1"], ]
   r c grain straw gsr in.north
470 10 24 2.84 4.1 0.69268 TRUE
```

Q98: Which are the plots with the largest positive and negative scores for PC1? How is this explained? Jump to A98

If you have looked closely at the biplot, you have surely noticed that the lower and left axes, which are related to the PC scores, do not correspond exactly to the scores (read the values for PC1 off the graph for the highest/lowest scores discovered just above). This is because the default biplot is scaled
by the standard deviation of each PC, i.e., its eigenvalue. If we reverse the scaling, we see the value on the plot:

```r
> pc$sdev
[1] 1.31521 0.51983
> pc$x[ix.max, ]/pc$sdev
   PC1      PC2
2.77685 -0.16547
> pc$x[ix.min, ]/pc$sdev
   PC1      PC2
-2.74588 -0.36596
```

Now you can properly locate the PC scores for field plots 79 and 470 on the biplot.

---

**Q99**: **What is the interpretation of the second PC?**  
Jump to A99

To help answer this question we can again look at high- and low-scoring observations, but this time for PC2.

```r
> mhw[ix.max <- which.max(pc$x[, "PC2"])]
   r c grain straw gsr in.north
15 15 1 3.46 8.85 0.39096 FALSE
> pc$x[ix.max, ]/pc$sdev
   PC1      PC2
0.82436 4.98651
> mhw[ix.min <- which.min(pc$x[, "PC2"])]
   r c grain straw gsr in.north
184 4 10 4.59 5.41 0.84843 TRUE
> pc$x[ix.min, ]/pc$sdev
   PC1      PC2
0.091197 -3.576666
```

---

**Q100**: **Interpret the two PCs and the proportion of variance explained by each in terms of this experiment.**  
Jump to A100

So far we have only looked at the observations in PC space, shown by the bottom (PC1) and left (PC2) axes. There is another set of axes on the graph, and vector arrows (drawn in red). The top (PC1) and right (PC2) axes show the eigenvector components (“rotations”, “loadings”) for the original variables, multiplied by the standard deviation (eigenvalue) of the corresponding component. The vector arrows, beginning at the origin and ending
in labels, show the eigenvectors for the two original variables, scaled as just explained. The value is at the centre of the label.

```r
> pc$rotation[, 1] * pc$sdev[1]

grain straw
0.93 0.93

> pc$rotation[, 2] * pc$sdev[2]

grain straw
-0.36757 0.36757
```

The cosine of the angle between the vectors is proportional to their correlation.

**Q101:** What does the biplot reveal about the correlation between grain and straw yield?  
**Challenge:** Repeat the analysis with unstandardized principal components. Explain the differences with standardized PCA for the proportion of variance explained per component, the rotations, the biplot, and the interpretation of the two PC’s.

### 13.1 Answers

**A96:** PC1 explains 86.5% and PC2 13.5% of the total variance. That is, about 85% of the information is in the first component; this is a measure of the redundancy.  
**A97:** The first PC is made up of large contributions from the standardized variables representing grain yield and straw yield. This is interpreted as general plant size: high yielding-plants score higher on PC1.  
**A98:** The maximum score is from plot 79, the minimum from plot 470. The first has both very high grain and straw yield, the second both very low. The interpretation that PC1 represents overall yield is confirmed.  
**A99:** The second PC represents a contrast between grain and straw yield, independent of the overall yield level. Positive values have higher-than-average straw yields, at a given level of grain yield. In other words, the grain-to-straw ratio (GSR) is low. The highest– and lowest– scoring plots for PC2 confirm this: plot 15 has a very low GSR: 0.391, whereas plot 184 has a very high GSR: 0.848.  
**A100:** In summary, the first PC, accounting for about 85% of the variance in both grain and straw yield, represents the overall yield level, whereas the second PC, accounting for about 15% of the variance, represents the grain/straw ratio, i.e.,
plant morphology independent of yield. The great majority of the overall variability in this field is due to variable yield. However, there is still a fair amount of variation in plant morphology that does not depend on overall plant size. Since this is one variety of wheat (i.e., no genetic variation) and one management, we conclude that local environmental factors affect not only yield but also plant morphology. Return to Q100 •

\[ \text{A101: The angle between the vectors representing the two yields is small, showing that the yields are highly-correlated.} \quad \text{Return to Q101 •} \]

14 Model evaluation

In §8, §9 and §12 models were evaluated by the goodness-of-fit (coefficient of determination, expressed as the adjusted \( R^2 \)) and compared by ANOVA. These are internal measures of model quality: the data used to evaluate the model is also used to build it.

Of course, the main use of a model is to predict; and what we really would like is some measure of the predictive success of a model, i.e., an external measures of model quality, using independent data to evaluate the model, not the data that was used to build it. The modelling steps so far were designed to best calibrate the model; now we want to evaluate it.

Note: What we call evaluation is often termed validation; we prefer the first term because we can never know if the model is valid, we can only evaluate how well it matches reality.

A common approach is to split the dataset into a calibration and an evaluation (often called a validation) set.

1. The model is developed using only the observations in the calibration set;
2. This model is used to predict at the observations in the evaluation set, using the actual (measured) values of the predictor (independent) variable(s);
3. These predicted values are compared to the actual (measured) values of the response (dependent) variable in the evaluation set.

14.1 Splitting the dataset

Splitting the dataset has two constraints:

1. The calibration set must be large enough for reliable modelling;
2. The evaluation set must be large enough for reliable evaluation statistics.

A common split in a medium-size dataset such as this one is 3 to 1, i.e., 3/4 for calibration and 1/4 for evaluation.

The next issue is how to select observations for each set. This can be:
• **random**: select at random (without replacement); this requires no assumptions about the sequence of items in the dataset;

• **systematic**: select in sequence; this requires absence of serial correlation, i.e., that observations listed in sequence be independent;

• **stratified**: first divide the observations by some factor and then apply either a random or systematic sampling within each stratum, generally proportional to stratum size.

To decide which strategy to use, we need to know how the dataset is ordered, and if there are any useful stratifying factors.

The sequence of observations is given by the `row.names` function; to see just the first few we use the `head` function with the optional `n` argument to specify the number of items to view:

```r
> head(row.names(mhw), n = 10)
[1] "1" "2" "3" "4" "5" "6" "7" "8" "9" "10"
```

The observations correspond to these are:

```r
> head(mhw, n = 10)
  r c grain straw gsr in.north
1  1 1  3.63  6.37 0.56986  TRUE
2  2 1  4.07  6.24 0.65224  TRUE
3  3 1  4.51  7.05 0.63972  TRUE
4  4 1  3.90  6.91 0.56440  TRUE
5  5 1  3.63  5.93 0.61214  TRUE
6  6 1  3.16  5.32 0.59774  TRUE
7  7 1  3.18  5.32 0.59774  TRUE
8  8 1  3.42  5.32 0.61957  TRUE
9  9 1  3.97  6.03 0.65837  TRUE
10 10 1  3.40  5.66 0.60071  TRUE
```

**Q102**: What is the sequence of observations? Could there be serial correlation?  

**Jump to A102**

**Q103**: What are possible stratifying factors?  

**Jump to A103**

**Q104**: Which of these strategies is best to apply to the Mercer-Hall dataset, and why?  

**Jump to A104**

**Task 84**: Select a random sample of 3/4 of the 500 observations as the calibration set, and the rest as the evaluation set.  

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The `sample` function selects a random sample from a vector (here, the row numbers of the dataframe); the `size` argument gives the size of the sample and the `replace` argument is logical: should sampling be with replacement?

Although we know there are 500 observations, it’s more elegant to extract the number programatically with the `dim “dimensions of a matrix”` function.

Finally, we use `set.seed` so your results will be the same as ours; in practice you would not use this, instead accept the random seed. The results will be similar but different with each random sample.

**Note:** The argument 123 to `set.seed` is arbitrary, it has no meaning. Any number can be used, the only purpose is to get the same “random” result in the subsequent call to `sample`.

```r
> dim(mhw)
[1] 500 6
> (n <- dim(mhw)[1])
[1] 500
> set.seed(123)
> head(index.calib <- sort(sample(1:n, size = floor(n * + 3/4), replace = F)), n = 12)
[1] 1 2 3 4 5 6 7 11 12 13 14 15
> length(index.calib)
[1] 375
```

**Task 85:** Assign the remaining observations to the evaluation set.

The very useful `setdiff` function selects the subset of a set that is not in another subset:

```r
> head(index.valid <- setdiff(1:n, index.calib), n = 12)
[1] 8 9 10 18 20 25 29 30 32 33 37 40
> length(index.valid)
[1] 125
```

Check that the two subsets together equal the original set; this gives a chance to introduce the `union` function and the `setequal “are the sets equal?” logical function.

```r
> setequal(union(index.calib, index.valid), 1:n)
[1] TRUE
```
14.2 Developing the model

From §12.2 we know that the multivariate additive model (straw yield explained by the additive effect of grain yield and field half) is clearly best in this particular experiment. However, in any other situation the field would be different and so would any effect of field half. We’d like a measure of how good is a model to predict straw yield from grain yield in general. So we work with the single-predictor model.

Task 86: Re-fit this model on the calibration set.

```r
> cal.straw.grain <- lm(straw ~ grain, data = mhw, subset = index.calib)
> summary(cal.straw.grain)
```

Call:
`lm(formula = straw ~ grain, data = mhw, subset = index.calib)`

Residuals:
```
Min 1Q Median 3Q Max
-1.9642 -0.3478 0.0102 0.4028 3.0278
```

Coefficients:
```
Estimate Std. Error t value Pr(>|t|)
(Intercept) 1.0699 0.2688 3.98 8.3e-05 ***
grain 1.3735 0.0674 20.39 < 2e-16 ***
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Residual standard error: 0.609 on 373 degrees of freedom
Multiple R-squared: 0.527, Adjusted R-squared: 0.526
F-statistic: 416 on 1 and 373 DF, p-value: <2e-16

Q105: How well do the model coefficients agree with the model based on all observations? Jump to A105

To answer this, fit the model based on all observations, then compare the absolute and relative differences of the coefficients, extracted from the model with the `coef` (or, `coefficients`) function:

```r
> model.straw.grain <- lm(straw ~ grain, data=mhw)
> (coef(cal.straw.grain) - coef(model.straw.grain))
```

```
(Intercept) grain
 0.203574  -0.056992
```

```r
> (((coef(cal.straw.grain) - coef(model.straw.grain)) + /coef(model.straw.grain))*100

(Intercept) grain
 23.4998  -3.9841
```
14.3 Predicting at the evaluation observations

Now we have a model, it can be used to predict at the observations held out from calibration, i.e., the evaluation set.

**Task 87**: Predict the straw yield for the observation set.

The `predict.lm` function, which can also be called as `predict`, uses an object returned by `lm` to predict from a dataframe specified by the `newdata` argument. Here the data should be the rows (cases) of the `mhw` dataframe that are part of the evaluation set; these row numbers are in the `index.valid` vector.

```r
> pred <- predict.lm(cal.straw.grain, newdata = mhw[index.valid, ])
```

**Task 88**: Compare them with the actual yields in this set, both numerically and graphically.

For convenience, we first extract the vector of actual yields from the evaluation data frame:

```r
> actual <- mhw[index.valid, "straw"]
```

We now compare the numeric summaries and side-by-side histograms, using a common scales for correct visualization.

```r
> summary(pred); summary(actual)

Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    
4.82   6.03    6.33     6.43 6.84   8.12

Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    
4.10   5.95    6.33     6.52 7.08   8.85

> par(mfrow=c(1,2))
> hist(pred, main="", xlab="Predicted straw yields, lb / plot", + breaks=seq(4,9.2,by=0.4), freq=F, ylim=c(0,.8))
> hist(actual, main="", xlab="Actual straw yields, lb / plot", + breaks=seq(4,9.2,by=0.4), freq=F, ylim=c(0,.8))
> par(mfrow=c(1,1))
```
Q106: What are the major differences in the distributions of the modelled and actual evaluation observations?

14.4 Measures of model quality*

A systematic approach to model quality was presented by Gauch Jr. et al. [20]. This is based on a comparison of the model-based predictions and the measured values. This should be a 1:1 relation: each model prediction should equal the true value. Of course this will rarely be the case; the decomposition proposed by Gauch Jr. et al. [20] shows the source of the disagreement and allows interpretation.

Note: Another commonly-used approach was proposed by Kobayashi and Salam [26], who also wrote a letter contrasting their approach to Gauch’s [25]; a comprehensive review of model evaluation, citing many applications and interpretations, is presented by Bellocchi et al. [3].

Gauch Jr. et al. [20] distinguish:

MSD: Mean Squared Deviation. This shows how close, on average the prediction is to reality. Its square root is called the Root Mean Squared Error of Prediction (RMSEP), expressed in the same units as the modelled variable.

SB: Squared bias. This shows if the predictions are systematically higher or lower than reality.

NU: Non-unity slope. This shows if the relation between predicted and actual is proportional 1:1 throughout the range of values; if not, there is either an under-prediction at low values and corresponding over-prediction at high variables (slope > 1), or vice-versa (slope < 1).

LC: Lack of correlation. This shows how scattered are the predictions about the 1:1 line.
It is quite common to report the RMSEP; this indeed is the single number that tells how closely, on average, the model predicted the evaluation points; however the decomposition shows the reason(s) for lack of agreement.

Some notation:

- there are $n$ total evaluation observations;
- $y_i$ is the true (measured) value of evaluation observation $i$;
- $\hat{y}_i$ is the predicted value of evaluation observation $i$;
- the overbar $\bar{y}$ indicates the arithmetic mean of the $y_i$, etc.

Then MSD, SB, NU, and LC are computed as:

$$
MSD = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2
$$

$$
SB = (\bar{y} - \bar{y})^2
$$

$$
NU = (1 - b^2) \frac{1}{n} \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2
$$

$$
LC = (1 - r^2) \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2
$$

where:

- $b$ is the slope of the least-squares regression of actual values on the predicted values, i.e., $\sum y_i \hat{y}_i / \sum \hat{y}_i^2$; this is also called the **gain**.

- $r^2$ is the square of the correlation coefficient $r_{1:1}$ between actual and predicted, i.e., $(\sum y_i \hat{y}_i)^2 / (\sum y_i)^2 (\sum \hat{y}_i)^2$.

Thus NU is the non-unity slope in the regression of actual on predicted, scaled by the sums of squares of the predicted values of the evaluation observations, and LC is the lack of correlation in the regression of actual on predicted, scaled by the sums of squares of the actual values of the evaluation observations.

These have the relation:

$$
MSD = SB + NU + LC
$$

That is, the total evaluation error consists of bias, gain, and lack of correlation. These are distinct aspects of model quality and can be interpreted as **translation** (SB), **rotation** (NU), and **scatter** (LC), respectively:

**Translation**: The model systematically over- or under-predicts;

**Rotation**: The relation between actual and predicted value is non-linear, that is, not a constant relation throughout the range of values;

**Scatter**: The model is not precise.
If there is significant translation or rotation, this indicates the model form is not correct; we show an example of this below in §14.5. If there is significant scatter, this indicates that the model does not well-describe the system; perhaps there are missing factors (predictors) or perhaps the system has a lot of noise that can not be modelled. Thus by examining the model evaluation decomposition the analyst can decide on how to improve the model.

We begin by visualizing the predictive success of the model on a 1:1 plot. Note that by convention this plot has the actual (measured) value on the y-axis (dependent) and the predicted (modelled) value on the x-axis (independent). That is, it shows how the prediction predicts the actual.

**Task 89**: Plot the actual vs. predicted grain yields of the evaluation set, with a 1:1 line.

The plot:

```r
> plot(actual ~ pred, ylab="Actual", xlab="Predicted", asp=1,
+       main="Mercer-Hall trial, straw yield, lbs/plot",
+       xlim=c(4.5,9), ylim=c(4.5,9));
> abline(0,1); grid()
```

**Q107**: How well does the calibrated model predict the evaluation observations?  

14.4.1 MSD

**Task 90**: Compute the MSD and RMSEP.
Q108: How does the RMSEP compare to the RMSE (root mean squared error) in the calibration linear model? What is the practical interpretation of the RMSE?

The RMSE of the linear model is the mean of the squared residuals, extracted from the linear model with the residuals function:

\[ \text{rmse} <- \text{sqrt} \left( \text{mean} \left( \text{residuals(} \text{cal.straw.grain}\right)^2) \right) \]

\[ [1] 0.60742 \]

14.4.2 SB

Task 91: Compute the bias and its square (SB).

\[ (\text{valid.bias} <- \text{mean}(\text{pred}) - \text{mean}(\text{actual})) \]

\[ [1] -0.085874 \]

\[ (\text{valid.sb} <- \text{valid.bias}^2) \]

\[ [1] 0.0073743 \]

\[ \text{valid.sb/valid.msd * 100} \]

\[ [1] 1.828 \]

Q109: What is the bias? Is it positive or negative? What does this imply about the model? What proportion of the MSD is attributed to the SB?

14.4.3 NU

The next component of the error the non-unity slope (NU) factor. To compute this we first need to compute \( b \), the slope of the least-squares regression of actual values on the predicted values. This is an interesting number in its own right, the gain, which we hope will be 1. The gain can be directly with the least-squares formula \( \sum y_i \hat{y}_i / \sum \hat{y}_i^2 \), but in practice it’s easier to
use the \texttt{lm} function to fit the slope and then extract that coefficient with the \texttt{coef} function:

\begin{verbatim}
> regr.actual.pred <- lm(actual ~ pred)
> summary(regr.actual.pred)

Call:
\texttt{lm(formula = actual ~ pred)}

Residuals:
Min 1Q Median 3Q Max
-1.6928 -0.3468 -0.0084 0.3260 2.3841

Coefficients:
\begin{verbatim}
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.2390   0.6142  -2.02  0.046 *  
pred         1.2062   0.0950  12.69 <2e-16 ***
\end{verbatim}
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.623 on 123 degrees of freedom
Multiple R-squared: 0.567, Adjusted R-squared: 0.564
F-statistic: 161 on 1 and 123 DF, p-value: <2e-16
\end{verbatim}

\begin{verbatim}
> plot(actual ~ pred, ylab="Actual", xlab="Predicted", asp=1, +
+     main="Mercer-Hall trial, straw yield, lbs/plot", +
+     xlim=c(4.5,9), ylim=c(4.5,9));
> abline(regr.actual.pred, col="red")
> abline(0,1); grid()
> text(4.5, 8.5, paste("Gain:", round(coef(regr.actual.pred)[2], 2)), +
+     pos=4, col="red")
> legend(7.5, 5, c("1:1","regression"), lty=1, col=c("black","red"))
\end{verbatim}
Q110 : What is the gain? Is it greater than or less than 1? What does this say about the model?  

The model summary shows that the gain is significantly different from zero, which is no surprise. To assess it against a null hypothesis of $\beta = 1$, one method is to remove the 1:1 line from the regression and then compare the regression slope to zero. The 1:1 line is simply the line where actual yield equals predicted yield, so first we subtract the predicted from the actual; this would reduce an exact 1:1 slope to 0. Then we re-fit and summarize the model. We also visualize the hypothesized 0 slope and the actual negative gain.

```r
> regr.actual.pred.0 <- lm(I(actual - pred) ~ pred)
> summary(regr.actual.pred.0)
```

Call:  
\begin{verbatim}
    lm(formula = I(actual - pred) ~ pred)
\end{verbatim}

Residuals:

\begin{verbatim}
    Min     1Q   Median     3Q    Max
-1.6928 -0.3468  -0.0084  0.3260  2.3841
\end{verbatim}

Coefficients:

\begin{verbatim}
        Estimate Std. Error t value Pr(>|t|)
(Intercept)  -1.239    0.614  -2.02  0.046 * 
pred        0.206    0.095   2.17  0.032 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
\end{verbatim}

Residual standard error: 0.623 on 123 degrees of freedom 
Multiple R-squared: 0.0368, Adjusted R-squared: 0.0289 
F-statistic: 4.7 on 1 and 123 DF, p-value: 0.0322
The `I` "as-is" function is used here to specify that the expression `(actual-pred)` is an arithmetic operation, not a model formula. This is necessary because `-` is a formula operator, meaning to remove a term from a model. However here we want the `-` operator to have its usual (arithmetic) meaning.

The slope coefficient here is exactly 1 less than the slope coefficient from the original regression of actual vs. predicted, as expected:

```r
> coef(regr.actual.pred)[2] - coef(regr.actual.pred.0)[2]
pred
 1
```

**Q111 : Is the gain of the regression of actual vs. predicted significantly different from 1?**  
*Jump to A111*

**Task 93 :** Compute the non-unity slope (NU) factor.

First, extract the slope from the fitted linear model:

```r
> b <- coef(regr.actual.pred)[2]
> names(b) <- NULL
> print(b)
```
1.2059

Note: The `names` function retrieves the name attribute; here it was assigned the coefficient name `pred` (from the independent variable’s name) by the `lm` function. We don’t need the name, so we assign it `NULL`.

Then we use equation (14.3). The factor $1/n \cdot \sum_{i=1}^{n}(x_i - \bar{x})^2$ is the mean squared deviation of the predicted values from their mean:

```r
> (valid.msd.pred <- mean((pred - mean(pred))^2))
[1] 0.34355
```

and this is multiplied by the squared deviation of the slope of the regression of actual on predicted from 1:1:

```r
> (valid.nu <- (1 - b)^2 * valid.msd.pred)
[1] 0.014564
> valid.nu/valid.msd * 100
[1] 3.6101
```

Q112: What is the magnitude of the non-unity factor NU? What proportion of the MSD is attributed to the NU? Jump to A112

14.4.4 LC

The final aspect of model quality is the lack of correlation, LC.

Task 94: Compute LC.

We use equation (14.4). The factor $1/n \cdot \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the mean squared deviation of the measured values from their mean:

```r
> (valid.msd.actual <- mean((actual - mean(actual))^2))
[1] 0.88105
```

and then this is multiplied by the lack of fit in the regression of actual on predicted, $(1 - r^2)$. The fit is a field in the model summary; it can also be computed directly with `cor` function for this bivariate model:

```r
> (r2 <- summary(regr.actual.pred)$r.squared)
[1] 0.56703
> (r2 <- cor(actual, pred)^2)
[1] 0.56703
> (valid.lc <- (1 - r2) * valid.msd.actual)
```

135
Q113: What proportion of the MSD is attributed to LC? Jump to A113 •

Check that the three components add up to the MSD:

```r
> print(valid.msd - (valid.sb + valid.nu + valid.lc))
[1] 3.3307e-16
```

Yes, within rounding error.

Q114: What is the weak point of the model? Is the linear form justified? Jump to A114 •

Challenge: Repeat the above analysis for several more random selections (splits 3/4 – 1/4) of subsets (see §14.1). Collect the statistics for MSD and its components and summarize them (minimum, maximum, IQR, range, mean). How much do MSD and its components change? How reliable then is information from a single split of this size?

Challenge: Repeat the above analysis for another (or several more) split sizes, e.g., 1/20, 1/10, 1/5, 1/3, 1/2 evaluation sets. Again, take several selections at each split, and summarize them as in the previous challenge. Also summarize the calibrations (model coefficients and their standard errors). At what splits are the calibration and evaluation results (considered separately) least variable? For this dataset what is the best balance?

14.5 An inappropriate model form*

The main use of the evaluation analysis is to discover an inappropriate model form. We can see how this works by fitting such a model. The example we choose is a linear model without an intercept. That is, we force a zero grain yield to also have a zero straw yield. This was discussed above in §8.4, where its deficiencies in this case were revealed; here we analyze these with the model evaluation components of the previous section.

Q115: Would a plot with no grain yield necessarily have no straw yield? Jump to A115 •

Task 95: Fit a linear model of straw predicted by grain, without an intercept, to the evaluation subset, and summarize it.
By default models fit with `lm` include an intercept; to remove it use the `-` formula operator to remove the intercept, symbolized by the term 1.

```r
> cal.straw.grain.00 <- lm(straw ~ grain - 1, data = mhw,
+   subset = index.calib)
> summary(cal.straw.grain.00)

Call:
  lm(formula = straw ~ grain - 1, data = mhw, subset = index.calib)

Residuals:
   Min     1Q   Median     3Q    Max
-2.117  -0.362   0.045   0.380   3.176

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
grain 1.639770   0.00804  204.0  <2e-16 ***

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.621 on 374 degrees of freedom
Multiple R-squared: 0.991, Adjusted R-squared: 0.991
F-statistic: 4.16e+04 on 1 and 374 DF,  p-value: <2e-16
```

Task 96: Plot the straw vs. grain yield of the evaluation set, with the with-intercept and no-intercept models shown as lines; include the (0,0) point.

The graphing limits are specified with the `xlim` and `ylim` arguments to `plot`:

```r
> plot(straw ~ grain, data = mhw, subset = index.calib,
+   xlim = c(0, 6), ylim = c(0, 9))
> title("Mercer-Hall trial, calibration dataset")
> abline(cal.straw.grain, lty = 2)
> abline(cal.straw.grain.00, col = "red")
> grid()
> legend(4, 1, c("with intercept", "no intercept"), lty = c(2,
+   1), col = c("black", "red"))
> text(0, 2.5, paste("Slope: ", round(coef(cal.straw.grain)[2],
+   2)), pos = 4)
> text(1, 0.5, paste("Slope: ", round(coef(cal.straw.grain.00)[1],
+   2)), col = "red")
```
Q116: What is the effect of forcing the regression through $(0,0)$? Can you determine by eye which one fits better the observations?  Jump to A116

Task 97: Compute the predicted values for the evaluation observations, using the no-intercept model.

```
> pred <- predict.lm(cal.straw.grain.00, newdata = mhw[index.valid, ])
> summary(pred)

            Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
       4.48  5.92   6.28    6.40  6.89   8.41
```

Task 98: Plot the actual vs. predicted on a 1:1 graph, along with the 1:1 line and a regression of actual vs. predicted.

```
> regr.actual.pred.00 <- lm(actual ~ pred)
> plot(actual ~ pred, ylab="Actual", xlab="Predicted", asp=1,
+       main="Mercer-Hall trial, straw yield, lbs/plot",
+      xlim=c(4.5,9), ylim=c(4.5,9));
> abline(regr.actual.pred.00, col="red")
> abline(0,1); grid()
> text(4.5, 8.5, paste("Gain:",
+       round(coef(regr.actual.pred.00)[2], 2)),
+      pos=4, col="red")
> legend(7.5, 5, c("1:1","regression"), lty=1,
+       col=c("black","red"))
```
Task 99: Compute the evaluation statistics.

\[
\begin{align*}
\text{msd.00} & \triangleq \frac{1}{n} \sum (\text{actual} - \text{pred})^2 \\
\text{rmsep.00} & \triangleq \sqrt{\text{msd.00}} \\
\text{sb.00} & \triangleq (\frac{\text{mean}^{\prime}(\text{pred}) - \text{mean}^{\prime}(\text{actual}))^2}{\text{msd.00}} \\
\text{nu.00} & \triangleq \frac{(1 - (\text{cor}^{\prime}(\text{actual}, \text{pred}))^2) \times \text{msd.00}}{\text{msd.00}^{\prime}} \\
\end{align*}
\]

\[
\begin{align*}
> (\text{msd.00} \leftarrow \text{mean}((\text{actual} - \text{pred})^2)) \\
[1] 0.39495 \\
> (\text{rmsep.00} \leftarrow \sqrt{\text{msd.00}}) \\
[1] 0.62845 \\
> (\text{sb.00} \leftarrow (\text{mean}^{\prime}(\text{pred}) - \text{mean}^{\prime}(\text{actual}))^2) \\
[1] 0.013428 \\
> (\text{nu.00} \leftarrow ((1 - (\text{coef}^{\prime}(\text{regr.actual.pred.00})[2])^2) \times \text{mean}((\text{pred} - \text{mean}^{\prime}(\text{pred}))^2)) \\
\text{pred} \\
4.9761e-05 \\
> (\text{lc.00} \leftarrow (1 - \text{cor}^{\prime}(\text{actual}, \text{pred})^2) \times \text{mean}((\text{actual} - \text{mean}^{\prime}(\text{actual}))^2)) \\
[1] 0.38147 \\
\end{align*}
\]

Task 100: Compute the relative contribution of the model evaluation elements to the overall quality.

\[
\begin{align*}
> \text{sb.00/\text{msd.00}} \times 100 \\
[1] 3.4 \\
> \text{nu.00/\text{msd.00}} \times 100 \\
\end{align*}
\]
pred
0.012599
> lc.00/msd.00 * 100
[1] 96.587
> msd.00 - (sb.00 + nu.00 + lc.00)
pred
5.5511e-17

Q117: How do these statistics compare with those for the with-intercept model? Interpret them geometrically – what could be wrong with this model?

14.6 Answers

A102: By row, within each column in the field. Adjacent plots, i.e., adjacent observations in the data frame, may not be independent. Thus there could be serial correlation.

A103: Possible stratifying factors are field half (in.north), row, and column. But we saw in §9.3 that field half only explains a small amount of the variance, well under 10%; below in §17 we will see the same result for row and column.

A104: A random sampling is indicated: (1) there is no useful stratifying factor; (2) there may be serial autocorrelation.

A105: The coefficients agree fairly well; the percentage change from the full model to the calibration model is 23.5% for the intercept, -6.6% for the grain yield.

A106: The actual yields are spread over a wider range, especially the maximum, and less concentrated at the central value (about 6.5 lbs plot\(^{-1}\)).

A107: For the most part fairly well (points are bunched around the 1:1 line) but several points are very badly predicted. There does not appear to be any gain.

A108: The RMSEP of evaluation is 0.635 lbs plot\(^{-1}\); the residual mean square error of the model is 0.607 lbs plot\(^{-1}\). These are similar but in this evaluation the RMSEP is a somewhat higher. The RMSEP is the mean prediction error of straw
yield that we expect for the “same” experimental structure where we measure grain yield and use it to predict straw yield.

Return to Q108

A109: There is a small bias: -0.0859. This negative bias shows that the actual is a bit greater than the predicted, i.e., on average the model slightly under-predicts. But the squared bias SB is a very small proportion of the total error, 1.828%. This is not a weak point of the model.

Return to Q109

A110: The gain is greater than 1: 1.21. So, high straw yields are somewhat under-predicted, and low straw yields are somewhat over-predicted – the model fits well in the middle of the range, but smooths out the extreme highs and lows, predicting towards the mean. A Thus part of the model imprecision is due to rotation: there is

Return to Q110

A111: Yes, the probability that it would be a Type I error to reject the null hypothesis that the coefficient of the 1:1 model is in fact 1 (i.e., the coefficient of the 0:1 model is in fact 0) is 0.03, so we can safely reject the null hypothesis.

Return to Q111

A112: The non-unity factor NU is 0.01456352. This is a fairly small part of the MSD, namely 3.61%. This is not the weakest point of the model, although it is significant.

Return to Q112

A113: The lack of correlation LC is 0.3815; this is most of the MSD, namely 94.562% of it.

Return to Q113

A114: The major problem with the model is lack of correlation (LC), i.e., the prediction precision. There is almost no bias and little gain. Thus the linear model form is well-justified.

Return to Q114

A115: No, a plant could grow vegetatively but never flower or produce grain.

Return to Q115

A116: The line through (0,0) has a steeper slope to reach the “cloud” of points, whereas the line with intercept has a positive intercept and so can have a gentler slope. It seems difficult to tell by eye which better fits the point cloud.

Return to Q116

A117: The no-intercept model has higher overall error: its RMSEP is 0.628 compared to 0.635 for the full model. But, this is not due to lack of correlation (LC): these are 0.381 and 0.381 respectively, i.e., identical! This shows nicely the value of the decomposition – the problem with the no-intercept model is not its lack of precision, this is exactly the same as for the with-intercept model.

Both squared biases (SB): are quite small: 0.013428 (no-intercept) and 0.007374 (intercept). Thus neither systematically over- or under-predicts.
This means the problem with the no-intercept model is in the non-unity slope (NU): 4.976e-05 (no-intercept) vs. 0.01456352. In the no-intercept model this is now a recognizable proportion (0.013%) of the total error. This is interpreted as a rotation; this is in fact what happens when we forced the regression through the origin.

We are done with these models and some other variables, except for the main model of straw yield and its RMSEP (see §15), so clean up the workspace:

```r
> rm(n, index.valid, index.calib, actual)
> rm(cal.straw.grain, pred)
> rm(valid.msd, rmse)
> rm(regr.actual.pred, regr.actual.pred.0, valid.bias,
+ valid.sb, valid.lc, b, valid.nu, valid.msd.pred,
+ valid.msd.actual, r2)
> rm(cal.straw.grain.00, regr.actual.pred.00, msd.00, rmsep.00,
+ sb.00, nu.00, lc.00)
```

15 Cross-validation*

In §14 the predictive success of a model was evaluated by evaluation against an independent dataset. Since we only had one dataset (the 500 plots), we were forced to create this set by a random split into calibration and evaluation sets. There are several problems here:

1. We lose precision in the model, because it’s based on fewer observations;

2. The split is random, so that a different split (with the same proportions) would give different results.

An approach to evaluation that uses the insight of point (2) but retains precision is leave-one-out cross validation, abbreviated LOOCV. This is also known as leave-one-out jackknifing [11, 13], where our main interest is in the accuracy of the prediction.

Note: The term “cross-validation” is used consistently in texts, papers and computer programs, so we use the term; however we consider it a form of evaluation, as explained in the previous §14.

The concept is simple:

1. For each observation:
   (a) remove it from the dataset, i.e., “leave one out”;
   (b) compute the model parameters (e.g., slope and intercept of a simple linear regression);
   (c) use this model to predict at the left-out point;
   (d) calculate the prediction error for this one point.

2. Compute the evaluation statistics for the set of prediction errors (one for each observation), as in §14.4.
These evaluation statistics are assumed to apply to the single equation (parameterization) computed from all observations.

Task 101: Write a function to compute the LOOCV fits for the linear model of straw yield predicted by grain yield. This function should take as arguments: (1) a model to cross-validate and (2) the dataset on which to cross-validate it. It should return (1) the LOOCV predictions and (2) the coefficients for each LOOCV model fit.

We use the function command to define a function; this was explained in §8.3.1. We define two arguments: (1) the model form to be cross-validated, named model within the function; (2) the dataset where the model should be applied, named dset.

This is the most complex function we’ve defined; here are some points of interest:

- The function is defined with the function command and stored in a workspace object with a name of our choosing; this name is then used to call the function;
- At the end of the function we use the return function to return a list, build with the list function, of (1) the LOOCV predictions as a vector named pred, (2) the coefficients of each of the models built omitting one observation as a matrix with one column per model coefficient, named coef. The former is used for the cross-validation, the latter to evaluate the robustness of the model to any unusual single observations.
- Both of the returned variables must be initialized before the for loop, which will fill in the rows one-by-one as each observation is omitted.
- The colnames function is used to assign names to the columns of the coefficients vector, using the paste function to create a list of coefficient names of the correct length, depending on the number of model parameters.
- The type conversion function as.character convert as sequence of numbers into a character vector, suitable to be used in paste.
- The for flow control structure defines a for-loop: the first expression after the for is the list of indices which are used inside the loop. Here we specify the sequence 1: nrow(dset); this uses the : operator, shorthand for a continuous integer sequence which could also be specified with the seq command, i.e., seq(1, nrow(dset), by=1).
- The value from this sequence is assigned to variable which we name i. This is only defined in the loop, and is used to specify observation numbers, both for omitting a row from the dataframe, i.e., [-i,], and selecting just the one row, i.e., [i,]; both use the [] matrix selection operator.
The results of each model fit (prediction and coefficients) are stored in the initialized vectors, at the correct slot, again using the loop index $i$.

Here is the function:

```r
Xval <- function(model, dset) {
  pred <- rep(0, nrow(dset))
  n <- length(coefficients(model))
  coef <- matrix(0, nrow = nrow(dset), ncol = n)
  colnames(coef) <- paste("b", as.character(0:(n - 1)), sep = "")
  for (i in 1:nrow(dset)) {
    m <- lm(formula(model), data = dset[-i, ])
    pred[i] <- predict(m, newdata = dset[i, ])
    coef[i, ] <- coefficients(m)
  }
  return(list(pred = pred, coef = coef))
}
```

**Task 102**: Apply this function to the model of straw yield predicted by grain yield, and display the structure of the returned object.

```r
xval.fit <- Xval(model.straw.grain, mhw)
str(xval.fit)
```

```
List of 2
$ pred: num [1:500] 6.06 6.69 7.32 6.44 6.06 ...
$ coef: num [1:500, 1:2] 0.862 0.865 0.861 0.864 0.868 ...
  ..- attr(*, "dimnames")=List of 2
     ..$ : NULL
     ..$ : chr [1:2] "b0" "b1"
```

**Task 103**: Display the actual observations of straw yield against the LOOCV fits, on a 1:1 line.

```r
lim <- range(xval.fit$pred, mhw$straw)
plot(mhw$straw ~ xval.fit$pred, asp = 1, xlim = lim, + ylim = lim, xlab = "LOOCV prediction", ylab = "Actual")
abline(0, 1)
grid()
```
Task 104: Compute the cross-validation residuals, i.e., the actual straw yields less the LOOCV fits; summarize them and display as a histogram.

```r
> xval.res <- xval.fit$pred - mhw$straw
> summary(xval.res)

   Min. 1st Qu.  Median      Mean 3rd Qu.    Max.
-3.05000 -0.37500 -0.01090  -0.00009  0.35500  2.03000

> hist(xval.res, main = "LOOCV residuals")
```

---

Frequency

0
50
100
150

-3 -2 -1 0 1 2

xval.res

LOOCV residuals
Recall, the RMSEP is the square root of the sum of squared residuals divided by the number of observations.

**Q118**: What is the LOOCV RMSEP? How does this compare to the RMSE of the fit (the internal measure) and the independent evaluation RMSE?  

Recall, the single estimate of independent evaluation RMSE was computed in §14.4.

Here are (1) RMSEP from cross-validation; (2) RMSE from the internal fit; (3) RMSEP from the single evaluation:

\[
\begin{align*}
&> \sqrt{\text{sum}(xval\text{-res}^2)/\text{nrow}(mhw))} \\
&\quad \quad [1] \ 0.61597 \\
&> \sqrt{\text{sum}(\text{residuals(model.straw.grain)^2)/(model.straw.grain$\text{df.residual})}} \\
&\quad \quad [1] \ 0.61477 \\
&> \text{print(valid.rmsep)} \\
&\quad \quad [1] \ 0.63514
\end{align*}
\]

**Challenge**: Compute and interpret the measures of model quality developed in §14.4, i.e., the RMSEP broken down into bias, gain and scatter.

The function also returned the LOOCV model coefficients.

**Task 105**: Summarize the LOOCV model coefficients, and compare with the best-fit coefficients (using all observations).

\[
\begin{align*}
&> \text{summary(xval.fit$coef, digits = 5)} \\
&\quad \quad \begin{array}{ll}
\text{b0} & \text{b1} \\
\text{Min.} & 0.77899 & \text{Min.} & 1.4214 \\
1\text{st Qu.} & 0.86293 & 1\text{st Qu.} & 1.4296 \\
\text{Median} & 0.86674 & \text{Median} & 1.4304 \\
\text{Mean} & 0.86628 & \text{Mean} & 1.4305 \\
3\text{rd Qu.} & 0.87003 & 3\text{rd Qu.} & 1.4313 \\
\text{Max.} & 0.90307 & \text{Max.} & 1.4514
\end{array} \\
&> \text{coefficients(model.straw.grain)} \\
&\quad \quad \begin{array}{ll}
(\text{Intercept}) & \text{grain} \\
0.86628 & 1.43050
\end{array}
\end{align*}
\]

The optional digits argument to summary is used here to show more than the default three significant digits.

**Q119**: How consistent are the LOOCV coefficients? Which one varied more? Why?  

Jump to A119
15.1 Answers

A118: The LOOCV RMSEP is 0.616 lb. plot\(^{-1}\). The single-estimate evaluation RMSEP is 0.6351 lb. plot\(^{-1}\). The internal RMSE of the fit is 0.6148 lb. plot\(^{-1}\).

The single evaluation RMSEP is the most conservative; this is because the evaluation is based on a large number of observations and a single model. Adjusting the model for each omitted observation (LOOCV) reduces the RMSEP to just a bit greater than the internal estimate.

There is only one LOOCV estimate and only one internal fit estimate, whereas the single evaluation by splitting the dataset could be done many times, each with a different random split. This is similar to **K-fold cross-validation**, where K is the proportion of the observations omitted each time.

A119: The slopes are very consistent; that is, leaving out any one observation hardly changes it: the total range is 0.03 compared to the single “best” value 1.4305. The intercepts are less consistent: the total range is 0.1241 compared to the single “best” value 0.8663. This shows that leaving out one very high or very low straw yield can move the line up or down.

16 Spatial analysis

To this point we have only considered the wheat yields in feature space (also known as attribute or property space). For example, the grain and straw yields form a two-dimensional ‘space’. But we have ignored an additional piece of information: the relative location of the plots in the field. Mercer & Hall clearly stated that there could be hot spots or geographic trends, meaning that the plots are not necessarily spatially independent. We now investigate this.

16.1 Geographic visualisation

We begin by visualising the agricultural field:

Task 106: Make a correctly-scaled map of the plot locations in the field, showing the plot numbers.

The plots are rectangular (longer N-S than wide E-W), so that by plotting the 25 columns and 20 rows in a square (the shape of the field, and the default for a bivariate plot), we get a geometrically-correct map. However, the default plotting function is from low to high indices, so that row 1 would be plotted at the bottom, when in fact it is at the top. We can specify the axis with the ylim argument, reversing the row order:

```r
> with(mhw, plot(c, r, type = "n", xlab = "column", ylab = "row", +   ylim = c(20, 1), main = "Layout of the Mercer-Hall uniformity trial"))
> abline(v = 1:25, lty = 1, col = "darkgray")
> abline(h = 1:20, lty = 1, col = "darkgray")
> with(mhw, text(c, r, rownames(mhw), cex = 0.5))
```
The `xlab` and `ylab` graphics parameters are used to specify the axis names. Also, the type of plot is specified with the `type` graphics argument, here set to "n", meaning “no plot” (yet) – this just sets up the plot area and axes. The actual plotting is then done with the `text` function, using the `rownames` function to extract the plot number from the dataframe.

**Task 107**: Make a post-plot of the grain yield, i.e. a map of the plot locations with symbol size proportional to the data value.

```r
> with(mhw,
+   plot(c, r, pch=21, col="black", bg="lightblue", ylim=c(20,1),
+       xlab="column", ylab="row",
+       main="Mercer-Hall uniformity trial",
+       sub="Area of circles proportional to grain yield",
+       cex=2*grain/max(grain))
```
We can visualise this better by displaying each point in a colour ramp. First we classify the observations into octiles (eight groups) with the `cut` function, using the `quantile` function to compute the octiles.

```r
> (q8 <- quantile(mhw$grain, seq(0, 1, length = 9)))
0% 12.5% 25% 37.5% 50% 62.5% 75% 87.5% 100%
> grain.c <- cut(mhw$grain, q8, include.lowest = T, labels = F)
> sort(unique(grain.c))
[1] 1 2 3 4 5 6 7 8
```

So the 500 yields have been grouped into eight classes.

A **colour ramp** is a list of colours in some visually-meaningful sequence. One example is produced by the `terrain.colors` function; the colours are given as hexadecimal numbers from 00 (absence of colour) to FF (saturation with the colour), for the three primary colours Red, Green and Blue:

```r
> terrain.colors(8)

[1] "#00A600FF" "#3EBB00FF" "#8BD000FF" "#E6E600FF" "#E9BD3AFF"  
[6] "#ECB176FF" "#EFC2B3FF" "#F2F2F2FF"
```

For example, the final colour in this ramp is `#F2F2F2`, which is a dark gray: equal saturations of the three primaries, and each of these has `#F2/#FF`, i.e. 95% saturation:

```r
> 0xf2/0xff

[1] 0.94902
```

This example shows that hexadecimal numbers may be used in R; they are indicated with the prefix 0x.
Now we use this colour ramp, selecting the appropriate colour for each quantile: dark green is lowest, white is highest.

```r
> with(mhw,
+   plot(c, r, pch=20, cex=2, bg="lightblue", ylim=c(20,1),
+        xlab="column", ylab="row",
+        main="Mercer-Hall uniformity trial",
+        sub="Colour of circles from low yield (green) to high (gray)",
+        col=terrain.colors(8)[grain.c])
```

Another way to visualize the field is with a 3D plot using the `wireframe` graphics function of the `lattice` package. The optional `aspect` argument controls ratio between the horizontal axes, as well as the vertical exaggeration; the optional `screen` argument controls the viewing angle; this is a list of rotation from looking across the rows, and the rotation from vertical viewing. We use a different colour ramp, obtained by a call to the `bpy.colors` function of the `sp` package, which we load with `require`:

```r
> require(sp)
> plot(wireframe(grain ~ r + c, data=mhw, drape=T,
+    aspect=c(1,.2), col.regions=bpy.colors(128),
+    main="Grain yield, lb. per plot",
+    screen= c(z=30, x=-60),
+    xlab="N to S", ylab="W to E",
+    sub="Looking SE from NW corner of field"))
```
Q120: Does there appear to be local spatial dependence, i.e. similar values near each other? Does this appear to vary over the field? Jump to A120.

We view this from another perspective: from the S (so, W to the left, E to the right), from somewhat a lower viewing angle, and with less vertical exaggeration; this corresponds to figure 1 of McBratney and Webster [35]:

```r
> plot(wireframe(grain ~ r + c, data=mhw, drape=T,
+   aspect=c(1,.08), col.regions=bpy.colors(128),
+   main="Grain yield, lb. per plot",
+   screen= c(z=270, x=-75), zlab="",
+   xlab="N to S", ylab="W to E",
+   sub="Looking N from S end of field"))
```
16.2 Setting up a coordinate system

Analysis of the spatial structure requires metric coordinates, rather than row and column numbers, since the plots are not square.

Task 108: Determine the field size and the plot dimensions.

The original experiment was in English units, but we will use metres. So, we start with some conversions to get the dimensions of each plot:

First, some conversion factors; note that 1 ha = 10 000 m²:

```r
> ha2ac <- 2.471054
> ft2m <- 0.3048
> (field.area <- 10000/ha2ac)
[1] 4046.9
```

Then we divide the side of the 1-acre field evenly into 20 rows and 25 columns to obtain the dimensions in meters and the area in m²:

```r
> (plot.area <- field.area/500)
```

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Task 109: Compute the total length and width in metres, confirm they are equal (because the field is a square, and confirm that they multiply to 1 ha (4 045.9 m²).

> (tot.len <- plot.len * 20)
[1] 63.615
> (tot.wid <- plot.wid * 25)
[1] 63.615
> tot.len * tot.wid
[1] 4046.9
> rm(tot.len, tot.wid)

Task 110: Compute coördinates for the centre of each plot.

Coördinates are assigned from an arbitrary origin of (0, 0) at the SW corner of the field, so that the coördinates of the centre of plot [1, 1] are half the plot size in both directions:

> plot.wid/2
[1] 1.2723
> plot.len/2
[1] 1.5904

Now we build a data frame of coordinates; first with the seq function to make vectors of the midpoints of the E and N directions, respectively; and then with the expand.grid function to make a dataframe with one row per combination:

> nrow <- length(unique(mhw$r))
> ncol <- length(unique(mhw$c))
> sx <- seq(plot.wid/2, plot.wid/2 + (ncol - 1) * plot.wid, +    length = ncol)
> sy <- seq(plot.len/2 + (nrow - 1) * plot.len, plot.len/2, +    length = nrow)
> xy <- expand.grid(x = sx, y = sy)
> rm(nrow, ncol, sx, sy)

The sequence for the y-axis starts with the highest coordinate for row 1 (which is at the top of the plot).

We keep `plot.wid`, `plot.len`, and `plot.area` to be used later.

### 16.3 Loading add-in packages

For most of the spatial analysis we will use two add-in packages; these are representative of the hundreds which have been implemented by practising statisticians. Here we will use the `sp` package [39], which is a foundation for spatially-explicit analysis in R, and the `gstat` package [38], which is an R implementation of the `gstat` geostatistics program [40]. Both of these are extensively discussed and illustrated in the textbook “Applied Spatial Data Analysis with R” by Bivand et al. [5].

When R starts, a number of basic packages are loaded; we can see these with the `search` function. Additional packages may be loaded with the `library` or `require` functions; both of these ensure that the package isn’t already loaded.

---

**Task 111**: Load the `sp` and `gstat` packages, and also the `lattice` graphics package.

```
> require(sp); require(gstat); require(lattice)
```

---

### 16.4 Creating a spatially-explicit object

The `sp` package adds a number of spatial data types, i.e. new object classes.

---

**Task 112**: Copy the `mhw` dataframe to a new object and convert the copy to class `SpatialPointsDataFrame`.

```
> mhw.sp <- mhw
> coordinates(mhw.sp) <- xy
> summary(mhw.sp)

Object of class `SpatialPointsDataFrame`
Coordinates:
   min     max
x 1.2723 62.343
y 1.5904 62.025
Is projected: NA
```
proj4string : [NA]
Number of points: 500
Data attributes:

<table>
<thead>
<tr>
<th></th>
<th>r</th>
<th>c</th>
<th>grain</th>
<th>straw</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min.</td>
<td>1.00</td>
<td>1</td>
<td>2.73</td>
<td>4.10</td>
</tr>
<tr>
<td>1st Qu.</td>
<td>5.75</td>
<td>7</td>
<td>3.64</td>
<td>5.88</td>
</tr>
<tr>
<td>Median</td>
<td>10.50</td>
<td>13</td>
<td>3.94</td>
<td>6.36</td>
</tr>
<tr>
<td>Mean</td>
<td>10.50</td>
<td>13</td>
<td>3.95</td>
<td>6.51</td>
</tr>
<tr>
<td>3rd Qu.</td>
<td>15.25</td>
<td>19</td>
<td>4.27</td>
<td>7.17</td>
</tr>
<tr>
<td>Max.</td>
<td>20.00</td>
<td>25</td>
<td>5.16</td>
<td>8.85</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gsr</th>
<th>in.north</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min.</td>
<td>0.391</td>
</tr>
<tr>
<td>1st Qu.</td>
<td>0.574</td>
</tr>
<tr>
<td>Median</td>
<td>0.604</td>
</tr>
<tr>
<td>Mean</td>
<td>0.611</td>
</tr>
<tr>
<td>3rd Qu.</td>
<td>0.642</td>
</tr>
<tr>
<td>Max.</td>
<td>0.850</td>
</tr>
</tbody>
</table>

Q121: What is the data type of the mhw.sp object? What is the bounding box, i.e. limits of the the coördinates?

Now that we’ve built the spatial object, we can save it for later use in another session:

```r
> save(mhw.sp, file = "mhw_spatial.RData")
```

16.5 More geographic visualisation

Once an object is in a spatial class, the `spplot` function can be used to make a nicely-coloured post plot:

```r
> mhw.sp.pix <- as(mhw.sp,"SpatialPixelsDataFrame")
> f1 <- spplot(mhw.sp.pix, zcol="grain", cuts=8,
+   col.regions=bpy.colors(64),
+   main="Grain yield, lb. per plot", key.space="right")
> f2 <- spplot(mhw.sp.pix, zcol="straw", cuts=8,
+   col.regions=heat.colors(64),
+   main="Straw yield, lb. per plot", key.space="right")
> f3 <- spplot(mhw.sp.pix, zcol="gsr", cuts=8,
+   col.regions=terrain.colors(64),
+   main="Grain/Straw ratio", key.space="right")
```

Task 113: Plot the grain, straw, and their ratio, coloured by their octile.
> print(f1, split=c(1,1,2,2), more=T)
> print(f2, split=c(2,1,2,2), more=T)
> print(f3, split=c(1,2,2,2), more=F)
> rm(f1, f2, f3)

Grain yield, lb. per plot

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>3.5</td>
<td>4.0</td>
<td>4.5</td>
</tr>
<tr>
<td>5.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Straw yield, lb. per plot

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Grain/Straw ratio

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td></td>
</tr>
</tbody>
</table>

And we can make a postplot with both colour and size, although its value for visualization is questionable:

> print(spplot(mhw.sp, zcol="grain", pch=15,
+    cex=1.6*mhw$grain/max(mhw$grain), cuts=12,
+    col.regions=bpy.colors(64),
+    main="Grain yield, lb. per plot",
+    sub="Symbol size proportional to yield",
+    key.space="right"))
Challenge: Display grain and straw yields on adjacent plots, using a gray-scale colour ramp. (Hint: see §B). What are the advantages and disadvantages, for visualization, of using the gray scale vs. colour ramps?

16.6 Answers

A120: There are clear clusters of similar values, e.g. the patch of low values centred near (16, 18). Clusters seem more obvious in the north than the south.

A121: SpatialPointsDataFrame; the bounding box is 1.2723 to 62.3426 (W-E) and 1.5904 to 62.0245 (S-N)

17 Spatial structure

Now that we have a spatially-explicit object, we can examine it for its spatial structure. This can be of two kinds: a trend across the entire area or a local structure that does not depend on absolute location.

17.1 Spatial structure: trend

One possibility for spatial structure is a trend across the field.

Task 114: Explore whether there is any trend in grain yield by row or column.
One way to do this is to compute the row and column mean yields, and then sort them from lowest to highest with the `sort` function:

```r
> with(mhw, sort(by(grain, r, mean), decreasing = FALSE))

r
15 16 7 8 20 17 10 4 18 14
6 11 1 12 2 5 3 9 13 19
```

```r
> with(mhw, sort(by(grain, c, mean), d = F))

c
17 15 24 14 18 22 23 5 21 16
19 12 13 1 9 25 8 2 20 6
11 3 7 10 4
4.1125 4.2820 4.4630 4.5280 4.5410
```

**Q122:** Does there appear to be any trend or pattern in the sequence of row or column numbers?  

We can see both the means and variability with a grouped boxplot, first by row and then by column. The `xlim` argument is used to display the row boxplots in correct geographical order.

```r
> boxplot(grain ~ r, horizontal = T, data = mhw, xlim = c(20, + 1), ylab = "Row number", xlab = "Grain yield, lb. per plot")

> boxplot(grain ~ c, data = mhw, xlab = "Column number", ylab = "Grain yield, lb. per plot")
```

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Q123: Does there appear to be any pattern by row or column, either in the median yield or the variability with each row or column? [Jump to A123]

Although only the columns show a slight trend, there could be a trend not oriented with these.

Task 115: Compute a first-order trend surface of grain yield.

```r
> ts1 <- lm(mhw.sp$grain ~ coordinates(mhw.sp))
> summary(ts1)

Call:
  lm(formula = mhw.sp$grain ~ coordinates(mhw.sp))

Residuals:
     Min       1Q   Median       3Q      Max
-1.13520  -0.29360   0.00690   0.31400   1.17110

Coefficients:
                              Estimate Std. Error  t value Pr(>|t|)
(Intercept)                  3.731260   0.051850  71.964  < 2e-16 ***
coordinates(mhw.sp)x         -0.000664   0.001067  -0.621   0.530
coordinates(mhw.sp)y        0.007498    0.001068   7.017 7.2e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.438 on 497 degrees of freedom
Multiple R-squared:  0.0909,   Adjusted R-squared:  0.0873
F-statistic: 24.9 on 2 and 497 DF,  p-value: 5.14e-11
```
Q124: *Is there a significant trend? How much of the variance is explained?*

This is not a promising approach, so we remove the trend surface object from the workspace:

```r
> rm(ts1)
```

17.2 **Spatial structure: local**

There is only a very weak trend in grain yield; but are there hotspots?

---

**Task 116:** Compute and display the variogram of the grain yield.

We use the `variogram` function of the *gstat* package to analyze the local spatial structure. We also specify the optional `plot.numbers = T` argument to print the number of point-pairs next to the variogram values; the optional `width` argument to specify the bin size (here, the plot width), and the optional `cutoff` argument (by default it is 1/3 of the largest distance between point pairs); here it is 10 plot widths.

```r
> v <- variogram(grain ~ 1, mhw.sp,
+   cutoff=plot.wid*10, width=plot.wid)
> print(plot(v, plot.numbers=T))
```

---

Q125: *Describe the shape of the variogram. Is there evidence of local spatial structure? What is the approximate range of local spatial dependence, i.e. the separation at which the experimental variogram reaches its sill (maximum)?*

---

Q126: *Across how many adjacent plots is there expected to be some spatial dependence?*
We now try to fit a theoretical variogram model to this empirical variogram. There appear to be two structures:

1. a short-range to about 5 m with rapid increase in semivariance with separation; this section is difficult to model because of the few point-pairs in the closest bin;
2. a gradual increase to a sill at about 18 m.

The total sill appears to be about 0.20, of which the nugget is about 0.02, the first partial sill about 0.15 and the second about 0.03. We initialize the variogram with these parameters. We use the the `vgm` (specify a variogram) function twice, with the `add.to` argument the second time to combine variogram models.

```r
> (vm <- vgm(0.15, "Sph", 5, 0.02))

   model psill range
   1  Nug  0.02  0
   2  Sph  0.15  5

> (vm <- vgm(0.03, "Sph", 20, add.to = vm))

   model psill range
   1  Nug  0.02  0
   2  Sph  0.15  5
   3  Sph  0.03  20

> print(plot(v, model = vm, main = "Estimated variogram model"))
```

We then adjust the variogram with the `fit.variogram` function:

```r
> (vmf <- fit.variogram(v, vm))

   model  psill  range
   1  Nug 0.000000  0.0000
   2  Sph 0.166884  4.9488
   3  Sph 0.039522 20.8112
```
The fit has reduced the nugget to zero; so at each point (inside a plot) there should be, as theory predicts, no spatial dependence. The nested structure clearly suggests that most of the spatial variability is within the first 5 m, so grouping a few plots should greatly reduce the between-plot variability of an experiment (Mercer & Hall’s objective).

**Q127**: The zero nugget also implies no measurement error. Is that a valid assumption in this case? Why or why not? What should be the minimum nugget variance?

### 17.3 Absence of spatial structure*

In this section we show that spatial data does not necessarily have spatial structure. This also is a chance to investigate some of R’s facilities for simulation. We can see how this field might look if there were no spatial dependence, i.e. if the variation in yields analyzed in §7 were randomly distributed across the field. We do this by applying the `sample` function to the vector of yields, to take a sample of the size as the original vector (extracted with the `length` function) without replacement. This ensures that each recorded yield appears once in the new vector.

**Task 117**: Permute the vector of grain yields into a random order, compare to the original vector.

We do this several times to see the effect of randomization. The `head` function displays the first few elements of a vector; the `sort` method sorts them. The `set.seed` function ensures that your results match those presented here; of course you can experiment with other randomizations. We show the first few records of the samples, both unsorted and sorted, with the `head` function.
> set.seed(4502)
> head(mhw$grain)
> head(s1 <- sample(mhw$grain, length(mhw$grain), replace = FALSE))
[1] 4.06 2.97 4.47 4.06 4.20 4.29
> head(s2 <- sample(mhw$grain, length(mhw$grain), replace = FALSE))
[1] 3.05 4.27 3.92 4.46 3.55 4.49
> head(s3 <- sample(mhw$grain, length(mhw$grain), replace = FALSE))
[1] 3.42 3.46 5.07 3.67 4.10 4.46
> head(s1)
[1] 4.06 2.97 4.47 4.06 4.20 4.29
> head(s2)
[1] 3.05 4.27 3.92 4.46 3.55 4.49
> head(s3)
[1] 3.42 3.46 5.07 3.67 4.10 4.46

Q128: Do the permutations have the same elements as the original vector? What is different? Jump to A128

Task 118: Display the spatial pattern of the randomized yields.

> par(mfrow=c(2,2))
> plot(mhw$c, mhw$r, pch=20, cex=2, bg="lightblue", ylim=c(20,1),
+ xlab="column", ylab="row", main="Randomization 1",
+ col=terrain.colors(8)[cut(s1, q8, include.lowest=T, labels=F)])
> plot(mhw$c, mhw$r, pch=20, cex=2, bg="lightblue", ylim=c(20,1),
+ xlab="column", ylab="row", main="Randomization 2",
+ col=terrain.colors(8)[cut(s2, q8, include.lowest=T, labels=F)])
> plot(mhw$c, mhw$r, pch=20, cex=2, bg="lightblue", ylim=c(20,1),
+ xlab="column", ylab="row", main="Randomization 3",
+ col=terrain.colors(8)[cut(s3, q8, include.lowest=T, labels=F)])
> plot(mhw$c, mhw$r, pch=20, cex=2, bg="lightblue", ylim=c(20,1),
+ xlab="column", ylab="row", main="Actual spatial distribution",
+ col=terrain.colors(8)[grain.c])
> par(mfrow=c(1,1))
Q129: How do the randomizations differ from each other and the original spatial pattern? *Jump to A129*

It may be difficult to see that the randomizations have no spatial structure. So, we examine this with variograms, as in §17.2, after converting the simulated objects to spatial objects.

```r
> s1 <- data.frame(s1); coordinates(s1) <- xy
> s2 <- data.frame(s2); coordinates(s2) <- xy
> s3 <- data.frame(s3); coordinates(s3) <- xy
> pv <- plot(variogram(grain ~ 1, mhw.sp, cutoff=plot.wid*10,
+     width=plot.wid), main="Real")
> p1 <- plot(variogram(s1 ~ 1, s1, cutoff=plot.wid*10,
+     width=plot.wid), main="Simulation 1")
> p2 <- plot(variogram(s2 ~ 1, s2, cutoff=plot.wid*10,
+     width=plot.wid), main="Simulation 2")
> p3 <- plot(variogram(s3 ~ 1, s3, cutoff=plot.wid*10,
+     width=plot.wid), main="Simulation 3")

> print(p1, split = c(1, 1, 2, 2), more = T)
> print(p2, split = c(2, 1, 2, 2), more = T)
```
Q130: Are the variograms of simulated fields similar to each other? To the variogram of the actual spatial arrangement of plots? What is the evidence that the simulated fields (actual yields randomly assigned to plots) has no spatial dependence?

The conclusion from this section must be that spatial dependence is not always present! It must be verified by variogram analysis or similar (e.g. spatial autocorrelograms) or trend surface analysis.

Remove the temporary variables from the simulation displays:

```R
> rm(xy, q8, grain.c, s1, s2, s3, pv, p1, p2, p3)
```

### 17.4 Spatial structure of field halves*

In §9 we computed an indicator variable to show which half of the field each plot is in. In a spatial analysis we may now ask whether these two halves have different spatial structures.

**Task 119**: Separate the dataset into two halves, one for each field half. To get a suitable data structure we use the `split` function to create one object with a list of two data frames, one for each half. We then assign their coordinates.
> mhw.sp.ns <- split(as.data.frame(mhw.sp), mhw.sp$in.north)
> coordinates(mhw.sp.ns$T) <- ~x + y
> coordinates(mhw.sp.ns$F) <- ~x + y
> summary(mhw.sp.ns)

    Length Class      Mode
FALSE  8 data.frame list
TRUE  8 data.frame list
T  250 SpatialPointsDataFrame S4
F  250 SpatialPointsDataFrame S4

> summary(mhw.sp.ns$T)

Object of class SpatialPointsDataFrame
Coordinates:
  min     max
x 1.2723 62.343
y 1.5904 62.025
Is projected: NA
proj4string : [NA]
Number of points: 250
Data attributes:
     r        c   grain  straw
Min. :1.0 Min. : 1 Min. :2.78 Min. :4.10
1st Qu.:3.0 1st Qu.: 7 1st Qu.:3.66 1st Qu.:5.73
Median :5.5 Median :13 Median :3.97 Median :6.14
Mean :5.5 Mean :13 Mean :3.96 Mean :6.28
3rd Qu.:8.0 3rd Qu.:19 3rd Qu.:4.27 3rd Qu.:6.86
Max. :10.0 Max. :25 Max. :5.13 Max. :8.64
gsr   in.north
Min. :0.482 Mode:logical
1st Qu.:0.596 TRUE:250
Median :0.635 NA's:0
Mean :0.636
3rd Qu.:0.669
Max. :0.848

Task 120 :  Compute the variograms for each half, and plot these along with
the combined variogram.

We first compute the variograms for the two field halves; we already have
the variogram for the entire field.

> v.n <- variogram(grain ~ 1, mhw.sp.ns$T, cutoff = 30)
> v.s <- variogram(grain ~ 1, mhw.sp.ns$F, cutoff = 30)

We now compute the figures, but do not print them right away; instead we
store them as plotting objects:

> g.max = max(v$gamma, v.n$gamma, v.s$gamma)*1.2
> plot.vgm.all <- plot(v, plot.numbers=T,
                    + main="All", ylim=c(0,g.max))
> plot.vgm.N <- plot(v.n, plot.numbers=T,
                    + main="N half", ylim=c(0,g.max))
We compute a common vertical axis from the maximum value of all three variograms, so we can compare them side-by-side.

Now we print these on one screen, specifying their positions in the plot:

```r
> print(plot.vgm.all, split = c(1, 1, 3, 1), more = T)
> print(plot.vgm.N, split = c(2, 1, 3, 1), more = T)
> print(plot.vgm.S, split = c(3, 1, 3, 1), more = F)
```

We now try to model the half-field variograms, as we did for the whole field in the previous section. The half-field variograms do not seem to show the nested structure of the whole-field variogram.

```r
> (vmS <- vgm(0.14, "Sph", 20, 0.09))
model psill range
1  Nug  0.09  0
2  Sph  0.14  20

> (vmN <- vgm(0.08, "Sph", 13, 0.11))
model psill range
1  Nug  0.11  0
2  Sph  0.08  13

> (vmSf <- fit.variogram(v.s, vmN))
model psill range
1  Nug  0.076024  0.000
2  Sph  0.141284 13.787

> (vmNf <- fit.variogram(v.n, vmS))
model psill range
1  Nug  0.112956  0.000
2  Sph  0.081287 14.747
```
> plot.vgm.all <- plot(v, plot.numbers = T, main = "All",
+   model = vmf, ylim = c(0, g.max))
> plot.vgm.N <- plot(v.n, plot.numbers = T, main = "N half",
+   model = vmNf, ylim = c(0, g.max))
> plot.vgm.S <- plot(v.s, plot.numbers = T, main = "S half",
+   model = vmSf, ylim = c(0, g.max))

> print(plot.vgm.all, split = c(1, 1, 3, 1), more = T)
> print(plot.vgm.N, split = c(2, 1, 3, 1), more = T)
> print(plot.vgm.S, split = c(3, 1, 3, 1), more = F)

Remove the plotting objects and scale:
> rm(g.max, plot.vgm.all, plot.vgm.N, plot.vgm.S)

**Q131**: Do the two halves appear to have different local spatial structure?  
*Jump to A131* •

Remove the variograms and models from the workspace:
> rm(v, v.n, v.s, vm, vmf, vmN, vmNf, vmS, vmSf)

We are also done with the field halves:
> rm(mhw.sp.ns)

**Challenge**: Repeat the analysis of this section with E-W field halves, instead of N-S field halves. Do you reach similar conclusions about the differences between the spatial structure of the field halves?

### 17.5 Answers

**A122**: The row and column numbers don’t seem to show any pattern or trend.  
*Return to Q122* •
A123: Although there are differences among rows and columns, there does not appear to be a trend. The higher-numbered columns (East half of the field) appear to be slightly lower (as a group) than the lower-numbered columns. There appears to be some short-range periodicity at a one-plot range (high followed by low) in both dimensions, although this is not regular. Return to Q123 •

A124: Only about 9% of the variance is explained; only the x-coordinate (E-W, across the columns from low- to high-numbered) is significant; it shows a slight trend towards the East; this agrees with the by-column boxplot. This trend was also noted by Patankar [37]. Return to Q124 •

A125: There is evidence of spatial structure: the semivariance increases up to about 13 m separation; the semivariance then fluctuates around a total sill of about $\gamma = 0.21 \text{lb}^2$. There appears to be a “nugget” effect (semivariance at zero separation) of about $\gamma = 0.05 \text{lb}^2$. Return to Q125 •

A126: Plot size is (3.18 m long x 2.55 m wide) ($\S$A), so the range of about 13 m corresponds to about four adjacent plots column-wise and five adjacent plots row-wise. Return to Q126 •

A127: Strictly speaking, this is not a valid assumption: we know from the description of the experimental protocol that the measurement precision was 0.01 lb ($\S$A), so the minimum nugget should be 0.0001 lb$^2$. However, this is very close to zero, in relation to the total sill of about 0.20 lb$^2$. This reported precision assumes that all other operations were carried out perfectly: the plot was exactly delineated, all grain and straw in the plot was harvested, the air-drying brought all samples to the same moisture level, and the hand-threshing did not lose any grain. The zero fitted nugget suggests that the experimental protocol was very carefully carried out. Return to Q127 •

A128: The elements of the permuted and original vectors are the same, all that changes is their sequence. Return to Q128 •

A129: The randomizations have a similar pattern but different locations of high and low values; there is no apparent spatial pattern. The actual spatial distribution shows clear concentrations of high and low values (“hot” and “cold” spots) with a spatial dependence of about 4 plots. Return to Q129 •

A130: The three simulations have very similar variograms: they all fluctuate around the sill (representing the total variability in the field), which is the same as for the variogram of actual data. This latter is quite different from the simulations, and shows clear spatial structure up to 15 m.

The lack of spatial dependence in the “random assignment” fields is proven by the variograms: a pure nugget effect, where the nugget variance is the same as the sill. Return to Q130 •
**A131** : There is a big difference between the structures in the two halves. The S half is more variable overall (higher total sill, about $\gamma = 0.22\text{lb}^2$), with a longer range around 20 m; the N half reaches a lower total sill (about $\gamma = 0.19\text{lb}^2$) at a shorter range, about 12 m. Both have a nugget effect, about $\gamma = 0.075\text{lb}^2$ in the S and $\gamma = 0.011\text{lb}^2$ in the N.

18 Generalized least squares regression*

In §8.2 we computed a relation of straw yield modelled from grain yield, which could then be applied to predict the straw yield of any plot, under similar conditions, with a measured grain yield:

```r
> coef(model.straw.grain <- lm(straw ~ grain, data = mhw.sp))
```

```
(Intercept) grain
  0.86628  1.43050
```

Clearly, however, this relation is not the same at each plot: that’s why there are non-zero residuals from the linear regression. But is there a spatial relation here? That is, are the residuals from the regression spatially correlated? If they are, they violate one of the assumptions of ordinary least-squares (OLS) linear regression, namely, the independence of and identical distribution (i.i.d) of the residuals. In that case, the solution by OLS presented in §8.2.2 is not valid and must be modified.

So, we now determine whether there is any evidence of non-independence of the residuals due to spatial correlation. The first step is visualization; afterwards we will model the suspected spatial correlation.

---

**Task 121** : Add the OLS model residuals to the spatial points data frame and show as a post-plot.

Again, as in §16.5, we also make a pixel version for easier visualization.

```r
> mhw.sp/msg.res <- residuals(model.straw.grain)
> mhw.sp.pix <- as(mhw.sp,"SpatialPixelsDataFrame")
> spplot(mhw.sp.pix, zcol="msg.res", col.regions=bpy.colors(64),
+ main="Linear model residuals",
+ sub="straw ~ grain, lb. per plot")
```
Q132: Does there appear to be spatial correlation among the residuals? At what scale? 

The post-plot suggests that the residuals are spatially-correlated. In §17.2 we saw how to reveal spatial structure of a variable with a variogram; here we apply that method to the residuals, to see if they are spatially-correlated.

Task 122: Compute and display their empirical variogram.

As in §17.2 we compute the variogram to a radius of 10 plot widths, with bins of plot width size:

```r
> vr <- variogram(msg.res ~ 1, loc = mhw.sp, cutoff = plot.wid * 10, width = plot.wid)
> plot(vr, pl = T, pch = 20, cex = 1.5)
```
Q133:

(1) How does the empirical variogram support the inference from the post-plot that there is spatial correlation among the residuals?

(2) Approximately how much of the residual variance is spatially-correlated at the shortest separation?

(3) What is the approximate range of spatial correlation? Jump to A133

Task 123: Model the variogram.

```r
> (vgmr <- fit.variogram(vr, model = vgm(0.15, "Sph", 20, + 0.05)))

model  psill  range
1  Nug 0.16901 0.0000
2  Sph 0.19183 7.4714

> plot(vr, model = vgmr, pch = 20, cex = 1.5)
```
Q134: Describe the spatial structure of the model residuals. Is this evidence of spatial correlation?

We conclude that the model residuals are not independent – they are spatially correlated. Although the regression coefficients computed in §8.2 are unbiased, the standard errors are too small and so the significance is overestimated, since there are effectively fewer degrees of freedom (plots partially duplicate each other’s information). Further, the coefficients may not be optimal.

The key difference here is that in the linear model, the residuals $\varepsilon$ are independently and identically distributed with the same variance $\sigma^2$:

$$y = X\beta + \varepsilon, \varepsilon \sim \mathcal{N}(0, \sigma^2 I)$$  \hspace{1cm} (18.1)

Whereas, now the residuals are considered themselves a random variable $\eta$ that has a covariance structure:

$$y = X\beta + \eta, \eta \sim \mathcal{N}(0, \mathbf{V})$$  \hspace{1cm} (18.2)

where $\mathbf{V}$ is a positive-definite variance-covariance matrix of the model residuals.

Continuing with the derivation from §8.2.2, Lark and Cullis [27, Appendix] point out that the issue here is that the error vectors can now not be assumed to be spherically distributed around the 0 expected value, but rather that error vectors in some directions are longer than in others. So, the measure of distance (the vector norm) is now a so-called “generalized” distance\textsuperscript{17}, taking into account the covariance between error vectors:

$$S = (y - X\beta)^T \mathbf{V}^{-1} (y - X\beta)$$  \hspace{1cm} (18.3)

Comparing this to the OLS equivalent (Equation 8.4), we see here the variance-covariance matrix of the residuals $\mathbf{V} = \sigma^2 \mathbf{C}$, where $\sigma^2$ is the variance of the residuals and $\mathbf{C}$ is the correlation matrix. This reduces to the OLS formulation of Equation 8.4 if there is no covariance, i.e., $\mathbf{V} = \mathbf{I}$.

\textsuperscript{17}This is closely related to the Mahalanobis distance
Expanding Equation 18.3, taking the partial derivative with respect to the parameters, setting equal to zero and solving we obtain:

\[
\frac{\partial}{\partial \beta} S = -2X^T V^{-1} y + 2X^T V^{-1} X \beta = 0
\]

\[
\hat{\beta}_{\text{GLS}} = (X^T V^{-1} X)^{-1} X^T V^{-1} y
\]

(18.4)

This reduces to the OLS estimate \( \hat{\beta}_{\text{OLS}} \) of Equation 8.6 if there is no covariance, i.e., \( V = I \).

In the case of spatial correlation, we ensure positive-definiteness (i.e., always a real-valued solution) by using an authorized covariance function \( C \) and assuming that the entries are completely determined by the vector distance between points \( x_i - x_j \):

\[
C_{i,j} = C(x_i - x_j)
\]

(18.5)

In this formulation \( C \) has a three-parameter vector \( \theta \), as does the corresponding variogram model: the range \( a \), the total sill \( \sigma^2 \), and the proportion of total sill due to pure error, not spatial correlation \( s^{18} \).

In modelling terminology, the coefficients \( \beta \) are called fixed effects, because their effect on the response variable is fixed once the parameters are known. By contrast the covariance parameters \( \eta \) are called random effects, because their effect on the response variable is stochastic, depending on a random variable with these parameters.

Models with the form of Equation 18.2 are called mixed models: some effects are fixed (here, the relation between the straw and grain yields) and others are random (here, the error variances) but follow a known structure; these models have many applications and are extensively discussed in Pinheiro and Bates [41]. Here the random effect \( \eta \) represents both the spatial structure of the residuals from the fixed-effects model, and the unexplainable (short-range) noise. This latter corresponds to the noise \( \sigma^2 \) of the linear model of Equation 18.1.

Q135: If \( s = 1 \), what does this imply?

To solve Equation 18.4 we first need to compute \( V \), i.e., estimate the variance parameters \( \theta = [\sigma^2, s, a] \), use these to compute \( C \) with equation 18.5 and from this \( V \), after which we can use equation 18.4 to estimate the fixed effects \( \beta \). But \( \theta \) is estimated from the residuals of the fixed-effects regression, which has not yet been computed. How can this “chicken-and-egg”\(^{19} \) computation be solved?

The answer is to use residual (sometimes called “restricted”) maximum likelihood (REML) to maximize the likelihood of the random effects \( \theta \) independently of the fixed effects \( \beta \). We continue with this in §18.2 below.

\(^{18}\) In variogram terms, this is the nugget variance \( c_0 \) as a proportion of the total sill \((c_0 + c_1)\).

\(^{19}\) from the question “which came first, the chicken or the egg?”
18.1 A detour into Maximum Likelihood*

To understand REML, we first explain the basics of ML, the **Maximum Likelihood** estimation, which is the derivation of the most “likely” values of model parameters, consistent with a set of observations.

There are three steps:

1. Specify a model with parameters.

   For example, the linear model with i.i.d. residuals of Equation 18.1:
   \[ y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I) \]. In the single-predictor case the parameter vector is \( \beta = (\beta_0, \beta_1) \), the intercept and slope of the linear model.

   All model forms are **assumptions**: we fit parameters given observations, but the model form is set by us. For this linear model, we are assuming that the observed values are the result of a linear deterministic process with coefficients \( \beta \), with stochastic errors with zero mean and a given variance, uncorrelated with each other. If this assumption is not valid, the rest of the analysis is invalid.

   **Note:** We can specify several models, fit their parameters, and compare their likelihoods.

We can rewrite the single-predictor linear model as the residuals expressed as the fits \( \epsilon = y - X\beta \). That is, for any chosen regression parameters \( \beta \), these are the errors \( \epsilon \sim \mathcal{N}(0, \sigma^2 I) \). Clearly, the we want to minimize the errors; we express this as the squared error \( (y - X\beta)^T(y - X\beta) \) to give equal weight to the positive and negative residuals.

Now, the **probability** of observing a specific response \( y_i \) from the random variable \( Y \), once \( \beta \) is fixed, is given by the normal probability of the associated residual, which we assume is normally-distributed with mean 0 and standard deviation \( \sigma \):

\[
\Pr(Y = y_i | \beta, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} (y_i - X_i\beta)^T(y_i - X_i\beta)}
\]  

(18.6)

This probability depends on the residual \( \epsilon_i = (y_i - X_i\beta) \), which can be directly calculated from the chosen regression parameters \( \beta \), but also on the standard deviation of the errors \( \sigma \), which must also be estimated from the observations.

The matrix product in the final term can be written as the squared residual:

\[
(y_i - X_i\beta)^T(y_i - X_i\beta) = (y_i - \beta_0 - \beta_1 x_i)^2 = \epsilon_i^2
\]

(18.7)

so Equation 18.6 can be written as:

\[
\Pr(Y = y_i | \beta, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} \epsilon_i^2}
\]

(18.8)
This probability can be computed with the `dnorm` function; here the mean residual is 0.

**Task 124**: For a fixed (assumed) value of the standard deviation, say the default 1, compute some probabilities of observed residuals.

```r
> dnorm(x = c(-1.96, -1, -0.5, 0, 0.5, 1, 1.96), mean = 0, 
   + sd = 1)
[1] 0.058441 0.241971 0.352065 0.398942 0.352065 0.241971 0.058441
```

**Task 125**: For a fixed (assumed) observed residual, say 1, compute the probability of observing that value, for several assumed values of the population standard deviation \( \sigma \).

```r
> s <- seq(0.4, 2, by = 0.2)
> data.frame(sd = s, p = dnorm(x = 1, mean = 0, sd = s))

<table>
<thead>
<tr>
<th>sd</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.403821</td>
</tr>
<tr>
<td>2</td>
<td>0.165795</td>
</tr>
<tr>
<td>3</td>
<td>0.228311</td>
</tr>
<tr>
<td>4</td>
<td>0.241971</td>
</tr>
<tr>
<td>5</td>
<td>0.234927</td>
</tr>
<tr>
<td>6</td>
<td>0.220797</td>
</tr>
<tr>
<td>7</td>
<td>0.205101</td>
</tr>
<tr>
<td>8</td>
<td>0.189940</td>
</tr>
<tr>
<td>9</td>
<td>0.176033</td>
</tr>
</tbody>
</table>
```

The probability of observing this fixed value of the residual (i.e., deviation from zero) increases until standard deviation 1 and then decreases, as the normal “bell-shaped” curve becomes flatter.

**Task 126**: Visualize the normal curves for the different standard deviations, and show the probability of the selected residual for each.

We can visualize this by applying the `curve` plotting function to values calculated with `dnorm`, with different standard deviations:

```r
> tmp <- rainbow(length(s))
> curve(dnorm(x, mean=0, s[i]), -3, 3, col=tmp[i], 
   + main="Normal probability density", 
   + sub="Varying the standard deviation", 
   + ylab="density",xlab="residual")
> for (i in 2:length(s))
>   curve(dnorm(x, mean=0, sd=s[i]), -3, 3, 
>      col=tmp[i], add=T)
> grid()
> abline(v=1, lty=2)
> legend(-3, i, s, lty=1, col=tmp)
```
2. Write an equation to compute the likelihood of observing the known values, given specific values of the parameters. This is the same conditional probability, but this time considering the observations \( y \) as fixed and the \( \beta \) as unknowns, to be solved for.

In the case where the observations are independent, the likelihood of the set of observations \( y \), given fixed \( \beta \), is defined as the product of their individual probabilities from Equation 18.6. This is because independent observations implies independent residual errors, so the joint probability of observing the actual vector \( y \) is the product of the probability of observing each one:

\[
L(\beta, \sigma^2 | y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{\varepsilon_i^2}{2\sigma^2}}
\]

\[
= \frac{1}{(2\pi \sigma^2)^{n/2}} \prod_{i=1}^{n} e^{-\frac{\varepsilon_i^2}{2\sigma^2}} \quad (18.9)
\]

This is now a function of the model parameters; as we vary these, the regression residuals \( \varepsilon \) change, so the the likelihood changes; when it is a maximum, these are the most probable values of the parameters. The variance \( \sigma^2 \) also can vary, and also affects the likelihood.

The likelihood function is usually written as a logarithm to allow easy differentiation; recall that \( \log(ab) = \log(a) + \log(b) \) and \( \log(a^c) = c \log(a) \), so that the product becomes a sum and an exponentiation becomes a product.

**Note:** The choice of parameters that maximizes the log-likelihood also maximizes the likelihood, because the logarithm is a monotonically increasing function.
Taking logarithms of both sides of Equation 18.9 we obtain:

\[
\ell(\beta, \sigma^2 | y) = \log \left( \frac{1}{(2\pi \sigma^2)^{n/2}} \right) - \frac{n}{2} \sum_{i=1}^{n} \frac{\varepsilon_i^2}{2\sigma^2} = \frac{-n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \varepsilon_i^2
\]

(18.10)

The first term depends only on the sample size; the second term depends on the sample size and the population variance; and the third term depends on the population variance and the regression coefficients (because they are used to compute \( \varepsilon_i \)).

3. Solve for the maximum value of the log-likelihood, either analytically or numerically.

Since this is an optimization problem, the obvious way to solve analytically is to differentiate Equation 18.10 with respect to each parameter and set to zero; then check if this is a maximum by the second derivative.

Note: Many likelihood functions cannot be solved analytically, in which case numerical optimization methods must be used, systematically varying the parameters, computing the likelihood, and looking for a maximum.

Here we have three partial derivatives, with respect to each of the three parameters to be estimated:

\[
\frac{\partial \ell}{\partial \beta_0} = \frac{1}{\sigma^2} \sum_i (y_i - \beta_0 - \beta_1 x_i) \quad (18.11)
\]

\[
\frac{\partial \ell}{\partial \beta_1} = \frac{x_i}{\sigma^2} \sum_i x_i (y_i - \beta_0 - \beta_1 x_i) \quad (18.12)
\]

\[
\frac{\partial \ell}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{n} \sum_i (y_i - \beta_0 - \beta_1 x_i)^2 \quad (18.13)
\]

Setting these equal to zero and simplifying:

\[
\sum_i (y_i - \beta_0 - \beta_1 x_i) = 0 \quad (18.14)
\]

\[
\sum_i x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \quad (18.15)
\]

\[
\sum_i (y_i - \beta_0 - \beta_1 x_i)^2 = n\sigma^2 \quad (18.16)
\]

Solving these three equations, we obtain the familiar least-squares estimators for the slope \( \beta_1 \) and the intercept \( \beta_0 \):
\[ \hat{\beta}_{1,ML} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} = \frac{S_{XY}}{S_{XX}} \quad (18.17) \]
\[ \hat{\beta}_{0,ML} = \bar{y} - \hat{\beta}_{1,ML} \bar{x} \quad (18.18) \]

The most likely variance of the residuals is:

\[ \hat{\sigma}_{ML}^2 = \frac{1}{n} \sum_i e_i^2 = \frac{1}{n} \sum_i (y_i - \beta_0 - \beta_1 x_i)^2 \quad (18.19) \]

### 18.1.1 Numerical solution

In this case the solution can be found analytically; in general a numerical solution is required. For illustration we show a brute-force approach, trying various values of the three parameters and then finding the maximum log-likelihood. In practical computations, gradient methods are used, e.g., a multivariate version of Newton-Raphson root finding. In complicated problems the likelihood surface may not be convex and there is a danger of finding only a local maximum.

**Task 127**: Write a function to compute log-likelihood for simple linear regression according to Equation 18.10; its arguments should be the parameter values and the observations (both predictor and predictand).

```r
> like <- function(beta0, beta1, sigma, x, y) {
+   s2 <- sigma^2
+   n <- length(y)
+   pred <- beta0 + beta1 * x
+   loglike <- -(n/2) * (log(2 * pi)) - (n/2) * (log(s2)) -
+      (1/(2 * s2)) * (sum((y - pred)^2))
+   return(loglike)
+ }
```

**Task 128**: Set up arrays of possible parameter values to be tested for their likelihood.

These should be based on guesses from the scatterplot or previous information; here we use the results of the linear model fit and vary them by ±10%:

```r
> coefficients(lm(straw ~ grain, data = mhw))

(Intercept)    grain
   0.86628     1.43050

> summary(lm(straw ~ grain, data = mhw))$sigma

[1] 0.61477
```
We create the three vectors and then all possible combinations, using the `expand.grid` function, and add a placeholder column for the computed log-likelihood:

```r
> coef <- round(coefficients(lm(straw ~ grain, data = mhw)), + 5)
> (beta0 <- coef[1] * seq(0.9, 1.1, by = 0.02))
[1] 0.77965 0.79698 0.81430 0.83163 0.84895 0.86628 0.88361 0.90093 
[9] 0.91826 0.93558 0.95291
> (beta1 <- coef[2] * seq(0.9, 1.1, by = 0.02))
[1] 1.2875 1.3161 1.3447 1.3733 1.4019 1.4305 1.4591 1.4877 1.5163 
[9] 1.5449 1.5736
> (sigma <- round(summary(lm(straw ~ grain, data = mhw))$sigma, + 5) * seq(0.9, 1.1, by = 0.02))
[1] 0.55329 0.56559 0.57788 0.59018 0.60247 0.61477 0.62707 0.63936 
[9] 0.65166 0.66395 0.67625
> beta <- expand.grid(beta0 = beta0, beta1 = beta1, sigma = sigma)
> beta$loglik <- 0
> dim(beta)
[1] 1331 4
> rm(coef, beta0, beta1, sigma)
```

**Task 129**: Compute the log-likelihood for each combination of parameters, given the observed wheat yield data, and find the maximum.

This is a “brute-force” computation of all combinations, using a `for` loop and then finding the maximum with the `which.max` function:

```r
> for (i in 1:length(beta$loglik)) beta$loglik[i] <- like(beta[i, + "beta0"], beta[i, "beta1"], beta[i, "sigma"], mhw$grain, + mhw$straw)
> summary(beta$loglik)

    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    
-821   -614    -530    -556   -484   -465
> (beta.mle <- beta[which.max(beta$loglik), ])

  beta0  beta1    sigma  loglik
666 0.86628 1.4305  0.61477  -465.22
```

As expected, these optimal parameter values are identical to those computed by `lm` using least squares.

**Task 130**: Visualize the likelihood surface, first 1D for each parameter separately and then 2D for a combination of two parameters.
First, the three per-parameter one-dimensional plots, holding the other two parameters constant:

```r
> par(mfrow = c(1, 3))
> tmp <- beta[(beta$sigma == beta.mle$sigma), ]
> tmp <- tmp[(tmp$beta1 == beta.mle$beta1), ]
> plot(tmp$loglik ~ tmp$beta0, type = "b", xlab = "intercept",
>       ylab = "log-likelihood", main = "Constant slope, s.d.")
> grid()
> abline(v = beta.mle$beta0)
> tmp <- beta[(beta$sigma == beta.mle$sigma), ]
> tmp <- tmp[(tmp$beta0 == beta.mle$beta0), ]
> plot(tmp$loglik ~ tmp$beta1, type = "b", xlab = "slope",
>       ylab = "log-likelihood", main = "Constant intercept, s.d.")
> grid()
> abline(v = beta.mle$beta1)
> tmp <- beta[(beta$beta1 == beta.mle$beta1), ]
> tmp <- tmp[(tmp$beta0 == beta.mle$beta0), ]
> plot(tmp$loglik ~ tmp$sigma, type = "b", xlab = "s.d.",
>       ylab = "log-likelihood", main = "Constant intercept, slope")
> grid()
> abline(v = beta.mle$sigma)
> par(mfrow = c(1, 3))
```

Second, a two-dimensional surface, holding one parameter constant, using the `wireframe` function of the `lattice` package:

```r
> tmp <- beta[(beta$sigma == beta.mle$sigma), ]
> wireframe(loglik ~ beta0 + beta1, data = tmp, aspect = c(1,
>       0.5), drape = T, main = "Log-likelihood, Constant s.d.")
```
18.2 Residual Maximum Likelihood

We first complete the theoretical derivation of REML (§18.2.1), and then show how to compute the GLS relation using this method (§18.2.2).

18.2.1 REML – theory

Returning now to Equation 18.2, where we can not assume i.i.d. residuals as in Equation 18.1, Lark and Cullis [27, Eq. 12] show that the likelihood of the parameters is now expanded to include the spatial dependence implicit in the variance-covariance matrix $V$, rather than a single residual variance $\sigma^2$. The log-likelihood is then:

$$\ell(\beta, \theta | y) = c - \frac{1}{2} \log |V| - \frac{1}{2} (y - X\beta)^T V^{-1} (y - X\beta)$$

(18.20)

where $c$ is a constant (and so does not vary with the parameters) and $V$ is built from the variance parameters $\theta$ and the distances between the observations. By assuming second-order stationarity\(^{20}\), the structure can be summarized by the covariance parameters $\theta = [\sigma^2, s, a]$, i.e., the total sill, nugget proportion, and range.

However, maximizing this likelihood for the random-effects covariance parameters $\theta$ also requires maximizing in terms of the fixed-effects regression parameters $\beta$, which in this context are called nuisance parameters since at this point we don’t care about their values; we will compute them after determining the covariance structure.

Both the covariance and the nuisance parameters $\beta$ must be estimated, it seems at the same time (“chicken and egg” problem) but in fact the technique of REML can be used to first estimate $\theta$ without having to know the nuisance parameters. Then we can use these to compute $C$ with equation 18.5 and

\(^{20}\) that is, the covariance structure is the same over the entire field, and only depends on the distance between pairs of points
from this $V$, after which we can use equation 18.4 to estimate the fixed effects $\beta$.

The maximum likelihood estimate of $\theta$ is thus called “restricted”, because it only estimates the covariance parameters (random effects). Conceptually, REML estimation of the covariance parameters $\theta$ is ML estimation of both these and the nuisance parameters $\beta$, with the later integrated out [41, §2.2.5]:

$$\ell(\theta|y) = \int \ell(\beta, \theta|y) \, d\beta$$  \hspace{1cm} (18.21)

Pinheiro and Bates [41, §2.2.5] show how this is achieved, given a likelihood function, by a change of variable to a statistic sufficient for $\beta$.

Lark and Cullis [27, Eq. 15], following Smyth and Verbyla [46], show that the log-likelihood of $\theta$ is then conditional on the sufficient statistic $t$ for $\beta$:

$$\ell(\theta|t) = c - \frac{1}{2} \log |V| - \frac{1}{2} \log |X^T V^{-1} X|$$  \hspace{1cm} (18.22)

$$- \frac{1}{2} y^T V^{-1} (I - Q) y$$

where $Q = X(X^T V^{-1} X)^{-1} X^T V^{-1}$

Since the nuisance parameters $\beta$ are not present in Equation 18.22, the likelihood of the covariance parameters $\theta$, which determine the variance-covariance matrix $V$, can be maximized independently of the nuisance (regression) parameters.

18.2.2 REML – computation

Equation 18.22 must be maximized by varying the three parameters in $\theta$ which determine $V$, there is no analytic solution. This is is a non-linear optimization problem over a large parameter space, and there is no guarantee of finding the true optimum. This is in contrast to ordinary least squares, which is a direct computation from the model matrix and observation values. This problem can be partially addressed by starting the solution with reasonable covariance parameters, for example, those inferred from a variogram fit to the OLS residuals. But even this is no guarantee; Lark and Cullis [27] used simulated annealing, which can escape local minima. We use a gradient method, a multivariate version of Newton-Raphson minimization.

In the R environment REML by the gradient method is implemented with the gls “Generalized Least Squares” function of the nlme “Linear and Non-linear Mixed Effects Models” package, based on the text by Pinheiro and Bates [41]. See Bates [2] for a simple introduction to the concepts, and how to use them in the R environment.

---

**Task 131**: Load the nlme package and examine the help for the gls function.

```R
> require(nlme)
> help(gls)
```
As with the \texttt{lme} function, this requires a formula (here called a ‘model’) and a dataframe in which to find variables. In addition, it requires either a correlation structure or a weights matrix. In our case we have a known spatial correlation, so we need to specify how the residuals may be correlated.

The \texttt{nlme} package provides a number of constructors of correlation structures for both time and spatial correlation. The spatial correlation models are similar to familiar variogram models: exponential, Gaussian, and Spherical (the structure we used to model the residuals, above). All that is required is a starting value for the range and nugget, which we extract from the fitted variogram model and put together in a list, and the formula for spatial coordinates. These constructors are functions named \texttt{corSpher}, \texttt{corExp} and so forth; see \texttt{?corClasses} for details.

A small complication is that \texttt{nlme} does not depend on the \texttt{sp} structures, so we need to convert to a dataframe with the \texttt{as.data.frame} function, so that the coordinate names \texttt{x} and \texttt{y} become visible.

\begin{description}
\item[TASK 132] Fit the regression model by GLS, using a spherical correlation structure based on the variogram analysis.
\end{description}

The model fitting takes a bit of time; we can see how much by enclosing the call to \texttt{gls} in the \texttt{system.time} function:

\begin{verbatim}
> system.time(
+   model.gls.straw.grain <-
+   gls(model=straw ~ grain,
+        data=as.data.frame(mhw.sp),
+        correlation=corSpher(
+          value=c(vgmr[2,"range"],vgmr[1,"psill"],
+          form="x+y", nugget=T))
+   )
+
+ user  system elapsed
9.267    0.162   9.459
\end{verbatim}

\begin{description}
\item[TASK 133] Summarize the model.
\end{description}

\begin{verbatim}
> summary(model.gls.straw.grain)

Generalized least squares fit by REML
Model: straw ~ grain
Data: as.data.frame(mhw.sp)
AIC  BIC  logLik
853.54 874.59  -421.77

Correlation Structure: Spherical spatial correlation
Formula: ~x + y
Parameter estimate(s):
   range  nugget
8.02312 0.37426
\end{verbatim}
Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Std.Error</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>1.5605</td>
<td>0.240165</td>
<td>6.4977</td>
<td>0</td>
</tr>
<tr>
<td>grain</td>
<td>1.2557</td>
<td>0.059545</td>
<td>21.0879</td>
<td>0</td>
</tr>
</tbody>
</table>

Correlation:

<table>
<thead>
<tr>
<th></th>
<th>(Intr)</th>
<th>grain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intr)</td>
<td>1.000</td>
<td>-0.978</td>
</tr>
</tbody>
</table>

Standardized residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.114035</td>
<td>-0.637046</td>
<td>-0.010328</td>
<td>0.619166</td>
<td>4.790892</td>
</tr>
</tbody>
</table>

Residual standard error: 0.61467
Degrees of freedom: 500 total; 498 residual

There is quite a bit of information in the model summary:

1. the fixed-effects model form;
2. the fitting information: Akaike’s Information Criterion (AIC), Bayes’ Information Criterion (BIC) and the log-likelihood of the final REML fit;
3. the random-effects model form, i.e., the correlation structure, with estimates of its parameters (here, range and proportional nugget);
4. the fitted model coefficients for the fixed effects, their standard errors and significance;
5. the correlation between parameters;
6. the residuals and residual standard error (i.e., lack of fit).

Q136:

(1) How do the GLS linear regression coefficients compare with those estimated by OLS?

(2) What parameters for the specified spatial correlation structure were fit by REML?

(3) How do these spatial structure parameters compare with the fitted variogram model? 

> coefficients(model.gls.straw.grain)

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Std.Error</th>
<th>t-value</th>
<th>p-value</th>
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<tr>
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<td>21.0879</td>
<td>0</td>
</tr>
</tbody>
</table>

> coefficients(model.straw.grain)

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Std.Error</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>0.86628</td>
<td>0.244131</td>
<td>3.5614</td>
<td>0.0004</td>
</tr>
<tr>
<td>grain</td>
<td>1.43050</td>
<td>0.244131</td>
<td>5.8527</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
We now have two models to compare; which is better? One way to answer this is to compute each model’s log-likelihood with the `logLik` “log-likelihood” function, which can be applied to most models; but this is not corrected for degrees of freedom. For that, we use the `AIC` “Akaike’s An Information Criterion” function; this is defined as:

$$AIC = -2 \log (\text{likelihood}) + 2 \, p$$  \hspace{1cm} (18.23)

where $p$ is the number of model parameters. Thus the AIC penalizes models with many parameters, similarly to the adjusted $R^2$ for linear models. Because of the change in sign, the lower AIC is better.

```r
> logLik(model.gls.straw.grain)
'log Lik.' -421.77 (df=5)
> logLik(model.straw.grain)
'log Lik.' -465.21 (df=3)
> AIC(model.gls.straw.grain)
[1] 853.54
> AIC(model.straw.grain)
[1] 936.43
```

This comparison can only be applied to hierarchical models, that is, where one is an extension of the other. In this case the GLS model has the same form for the regression parameters but an extended form for the residual structure, so they can be compared.

Q137 : Is the GLS model more likely than the OLS model? Which model has more parameters? Which model is better according to the AIC? Jump to A137 •

Finally, we visualize the effect on the regression model fit of using GLS instead of OLS.

Task 134 : Plot the straw vs. grain yield, with the OLS and GLS regression lines.

```r
> plot(straw ~ grain, data=mhw, pch=20,
+     sub="black: OLS; red: GLS")
> grid()
> abline(model.straw.grain)
> abline(model.gls.straw.grain, col="red")
```
Q138: What may account for the shallower slope of the GLS line? Jump to A138

**Challenge:** Recompute the GLS regression, but using an exponential variogram model form – this is often used for these sorts of problems, because it has a simple structure and interpretation.

You will first have to re-fit the residual variogram with an exponential model to obtain starting values for gls.

Compare the GLS regression line with that obtained with the spherical model; compare the fitted correlation structure with that from the spherical model. Plot all three lines on the scatterplot and comment on the differences. How much difference did the choice of correlation structure make in the estimates?

Compare the log-likelihoods of the two GLS models (exponential and spherical variogram forms); which is more likely?

**Note:** Recall that the reported range parameter of an exponential model is one-third of its effective range, i.e., when the correlation is reduced to 5% of the total.

### 18.3 Answers

**A132:** Yes, there are definitely many high (orange to yellow) and low (blue) patches of a few adjacent plots. Return to Q132

**A133:** (1) Yes, the variance is less at closer separations; (2) about (0.35 –
0.2)/0.35 \approx 40\% of the total sill appears to be structural (not nugget variance); 
(3) the range is about 6 to 7 m, i.e., about two to three plots. Return to Q133 •

A134 : There is a fairly high nugget (theoretically, the same as for the residuals of straw yield) but clear spatial structure to about 6 meters (a bit more than two plot widths) and then some fluctuation around a sill, suggestive of periodicity; see §21. Return to Q134 •

A135 : A proportion of pure noise $s = 1$ means that the nugget $c_0 = \sigma^2$, i.e., is equal to the total sill ($c_0 + c_1$ in variogram terms), so the residuals have no spatial correlation and the OLS solution is valid; there is no need to account for correlation in the residuals by GLS. Return to Q135 •

A136 : 
(1) The intercept is higher (1.56 vs. 0.87) and the slope is shallower (1.26 vs. 1.43).
(2) A range of just over 8 m and a nugget of about 0.37.
(3) The range is quite comparable to the variogram model fit, just a little longer; but the nugget has a different meaning. Here it is the proportion of the total sill, which for the variogram fit is 0.4684; the REML fit decreases this a bit. Return to Q136 •

A137 : The GLS model is substantially more likely than the OLS model; its log-likelihood is -421.8 compared to -465.2 for the OLS model. The GLS model has two more parameters, because three parameters are needed for the covariance structure, but only one for the variance assuming i.i.d. residuals. Still, the AIC for the GLS model, 853.5, is much superior to that for the OLS model, 936.4 Return to Q137 •

A138 : If the very high- and very low-yielding plots, with the highest leverage on the regression line, are spatially-correlated (which seems likely), the clusters in the scatterplot at extreme upper-right (above the average line) and extreme lower-left (below) will have less effect; because of spatial correlation they will be effectively clustered by the GLS fit, to some degree sharing their weights. So the line will not be pulled so much by these high-leverage plots. Return to Q138 •

19 Geographically-weighted regression*

Another approach to spatial correlation in regression is to use it to reveal an underlying process. In §8.2 and §18 we implicitly assumed that the process by which straw and grain yields are related (i.e., the plant physiology in response to environmental factors) is the same everywhere. The environmental factors could result in higher or lower yields (linear model intercept), but the relation of straw to grain is constant (linear model slope). But is this true?

A recently-developed technique for investigating this is “geographically-weighted regression” (GWR). This computes the coefficients of the regression equa-
tion for each plot, in a local window which moves across a spatial field. Further, it weights plots near to the target plot more than those further away, using a density kernel. Thus the equation reported at each point can be thought of as the local linear relation between the response and predictor, which is now allowed to vary across the field. The main interpretive interest in GWR is the spatial pattern of the coefficients, which is taken as evidence of a spatially-varying process.

GWR is comprehensively described by Fotheringham et al. [17] and briefly by Fotheringham et al. [16, §5.4.3]; these same authors maintain a web page\footnote{http://ncg.nuim.ie/ncg/GWR/whatis.htm} with tutorial material.

One implementation of GWR in the R environment is the \texttt{spgwr} package\footnote{Although the package loads with the somewhat of-putting disclaimer “NOTE: This package does not constitute approval of GWR as a method of spatial analysis”!}, which builds on the \texttt{sp} package loaded in §16.3.

---

**Task 135**: Load the \texttt{spgwr} package.

\begin{verbatim}
> library(spgwr)
\end{verbatim}

A key issue in GWR is the size and shape of the window to be moved over the field, and centred on each point in turn. This is closely related to kernel density estimation of a single variable (not a regression relation). The most common kernel shape is Gaussian (bivariate normal, the familiar bell-curve), and the bandwidth is often chosen to minimize the average cross-validation error of all the points predicted by their own local regression.

---

**Task 136**: Compute the optimal bandwidth for GWR of the straw vs. grain relation.

The \texttt{gwr.sel} function does this:

\begin{verbatim}
> (bw <- gwr.sel(straw ~ grain, data = mhw.sp, adapt = F, + verbose = F))
\end{verbatim}

[1] 7.7937

**Q139**: To the range of the variogram of which variable should this bandwidth correspond? \quad \textit{Jump to A139}

---

**Task 137**: Compute and model the empirical variogram of the grain/straw ratio.

\begin{verbatim}
> (vmf.gsr <- fit.variogram(v.gsr <- variogram(gsr ~ 1, loc=mhw.sp), + model=vgm(0.004, "Sph", 10, 0.002)))
\end{verbatim}
<table>
<thead>
<tr>
<th>model</th>
<th>psill</th>
<th>range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nug</td>
<td>0.0018 0.0000</td>
<td></td>
</tr>
<tr>
<td>Sph</td>
<td>0.0016 8.4351</td>
<td></td>
</tr>
</tbody>
</table>

Q140: Does the effective range of the fitted variogram model of the grain/straw ratio match the fitted bandwidth for GWR?  

Task 138: Compute the GWR for the straw vs. grain regression.

The `gwr.sg` function computes this:

```r
> (gwr.sg <- gwr(straw ~ grain, data = mhw.sp, bandwidth = bw))
Call:
gwr(formula = straw ~ grain, data = mhw.sp, bandwidth = bw)
Kernel function: gwr.Gauss
Fixed bandwidth: 7.7937
Summary of GWR coefficient estimates at data points:

               Min. 1st Qu.  Median 3rd Qu. Max.  Global
X.Intercept. -0.0923 1.1600  1.6200  2.0600  3.4400 0.87
grain         0.8950 1.1000  1.2500  1.4000  1.7100 1.43
```

Q141: How much do the local slopes vary? How do they compare with the slope computed for the relation over the whole field?

Task 139: Plot the GWR slopes (coefficient of straw vs. grain) across the field.

The `SDF` field of the fitted object contains the fitted coefficients, the prediction from the local model, the residuals, and the local goodness-of-fit. The `grain` field within the `SDF` field is the slope coefficient.

For easier visualization, we convert the points representing plots into pixels.

```r
> gwr.coef <- as(gwr.sg$SDF,"SpatialPixelsDataFrame")
> print(spplot(gwr.coef, zcol="grain",
+       col.regions=bpy.colors(64),
+       key.space="right", cuts=8,
+       main="Slope: straw ~ grain"))
```
Q142: Does the relation of straw vs. grain appear to vary across the field? What is the interpretation?

Task 140: Plot the GWR residuals and investigate if they have any spatial structure.

The `gwr.e` field within the SDF field contains the residuals.

```r
> print(applot(gwr.coef, zcol="gwr.e",
+ col.regions=bpy.colors(64),
+ key.space="right", cuts=8,
+ main="Slope: straw ~ grain")
```
To check for spatial structure, we compute the empirical variogram of these GWR residuals, and model it. A technical point: the variogram method can not handle non-square grid cells, so we have to convert to spatial points.

```r
> vr.gwr <- variogram(gwr.e ~ 1, loc = as(gwr.coef, "SpatialPointsDataFrame"))
> plot(vr.gwr, plot.numbers = TRUE, pch = 20)
```

Now fit it, using starting values estimated from the empirical variogram plot:

```r
> (vmf.r.gwr <- fit.variogram(vr.gwr, +                 model=vgm(0.05, "Sph", 8, 0.25)))

             model   psill   range
     1     Nug 0.17799 0.0000
     2     Sph 0.13926 7.7842
> plot(vr.gwr, plot.numbers=TRUE, pch=20, +      model=vmf.r.gwr)
```
**Task 141**: Plot the empirical variogram and fitted variogram model for these residuals, along with those from the OLS global fit (§8.2.1) and the REML fit (§18) to the entire dataset.

We first must compute and model the variogram for the REML fit:

```r
> mhw.sp$gls.res <- residuals(model.gls.straw.grain)
> vr.gls <- variogram(gls.res ~ 1, loc = mhw.sp, cutoff = plot.wid * 10, width = plot.wid)
> (vmf.r.gls <- fit.variogram(vr.gls, model = vgm(0.1, "Sph", 5, 0.2)))
```

<table>
<thead>
<tr>
<th>model</th>
<th>psill</th>
<th>range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nug</td>
<td>0.14957</td>
<td>0.0000</td>
</tr>
<tr>
<td>Sph</td>
<td>0.20439</td>
<td>7.2488</td>
</tr>
</tbody>
</table>

To put these on one plot it’s easiest to use the base graphics `plot` method to establish the plotting axes and show one of the point sets; we then use the `lines` and `points` functions to add more point sets and lines. The lines are computed from the fitted variogram models with the `variogramLine` function.

```r
> ylim.plot=c(0, max(vr.gwr$gamma, vr.gls$gamma, vr$gamma))
> plot(gamma ~ dist, data=vr.gwr, ylim=ylim.plot,
+      type="b", lty=2, col="red", xlab="separation, m",
+      ylab="semivariance, (lbs plot^{-1})^2",
+      main="Regression model residuals")
> lines(variogramLine(vmf.r.gwr, maxdist=max(vr.gwr$dist)),
+       col="red")
> points(gamma ~ dist, data=vr, type="b", lty=2, col="blue")
> lines(variogramLine(vgmr, maxdist=max(vr.gwr$dist)),
+       col="blue")
```

---

**Q143**: Is there any spatial structure in the GWR residuals?  

**Jump to A143**

---

**A143**: **Is there any spatial structure in the GWR residuals?**

We first must compute and model the variogram for the REML fit:

```r
> mhw.sp$gls.res <- residuals(model.gls.straw.grain)
> vr.gls <- variogram(gls.res ~ 1, loc = mhw.sp, cutoff = plot.wid * 10, width = plot.wid)
> (vmf.r.gls <- fit.variogram(vr.gls, model = vgm(0.1, "Sph", 5, 0.2)))
```

<table>
<thead>
<tr>
<th>model</th>
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Q144: Explain the change in spatial structure from (1) OLS residuals, (2) GLS residuals, (3) GWR residuals.

19.1 Answers

A139: In this case of a bivariate relation, the bandwidth might be expected to correspond to the variogram range for the grain/straw ratio, which also represents the same process as the straw vs. grain regression.

A140: The bandwidth 7.8 m is a bit less than the range for grain/straw with a spherical model, 8.4; but they are comparable.

A141: There is quite a wide range of both coefficients; the IQR is also wide. The global coefficients are nowhere near the medians of the GWR coefficient distributions.

A142: Most definitely. There is a very high straw yield per unit grain yield in the NW and SE, very low in the SW. The process by which carbohydrates are partitioned between grain and straw appears to be systematically different in patches of about 1/3 field size, not just random noise plot-to-plot.
A143: Yes, the fitted variogram model shows a range of 7.8 m and a structural sill to total sill ratio of 44%.  

A144: The OLS fit has the highest residuals; these are slightly lowered by the REML fit, presumably because it better accounted for very high and low values close to each other. The GWR residuals are considerably lower and are fit with a slightly longer range. The lower residuals are because the GWR fit is local and so can better adjust to local relations between straw and grain.  

20 Management zones*

In §16 we saw that there is spatial structure in the field, i.e., there are clusters of similar grain yields, straw yields, and grain/straw ratios. This observation is common for farmers, and they may attempt to divide a seemingly homogeneous field into management zones. For example, areas with a low grain/straw ratio may have excess N, leading to excessive vegetative growth, so the farmer may use less fertilizer in such a zone. These zones may be based on covariates that are not homogeneous across the field (e.g., slope, surface soil texture) or by on-the-go sensor measurements (e.g., electrical conductivity of the surface soil) but may also be inferred from a cluster analysis of grain yields [15, 32].

The concept is that clusters in feature (attribute) space may also be clustered in geographic space, leading to delineation of zones. In this section we take a simple approach to defining management zones in the Mercer & Hall field, mainly as an exercise in cluster analysis.

Let’s look again at the straw vs. grain scatterplot to see if we can estimate how many clusters might be distinguished in feature space. We take this chance to introduce the ggplot2 “grammer of graphics” [57] graphics package. This has a very different approach than base graphics towards building up a plot; see the textbook by the package author Wickham [56].

Task 142: Display a scatterplot of straw vs. grain, with the grain/straw ratio highlighted by a colour ramp.

```r
> library(ggplot2)
> ggplot(mhw, aes(grain, straw, col=gsr)) + geom_point()
```
Clearly there is a wide range of yields of both grain and straw, as well as a wide range of grain/straw ratios.

**Q145**: *How many clusters of similar yields and ratios do you see? How could these be described?*

**Jump to A145**

**Task 143**: Divide the plots into six feature-space clusters, based on the grain and straw yields.

The *k*-means algorithm [22] divides a multivariate dataset into an analyst-defined *k* groups such that the within-cluster sum of squares is minimized. It is implemented in the `kmeans` function. Since a true optimum is computationally unfeasible, the algorithm repeats for a user-defined `nstart` number of random starting cluster centroids, and compares the sum-of-squares achieved for each one, taking the least as the result.

```r
> (gs.cluster <- kmeans(mhw[, c("grain","straw")],
                          centers=6, nstart = 20))

K-means clustering with 6 clusters of sizes 104, 39, 64, 67, 128, 98

Cluster means:
   grain straw
1 3.7890 6.4134
2 3.2779 4.9644
3 4.5784 8.0356
4 4.1525 5.9930
5 4.1949 7.1593
6 3.5126 5.7612

Clustering vector:
[1] 1 4 5 5 6 6 2 6 4 6 6 5 5 5 3 5 5 1 1 5 5 3 3 3 1 6 6 6 1 1 1
```

196
Within cluster sum of squares by cluster:
(between_SS / total_SS = 87.6 %)

Available components:
[1] "cluster"   "centers"   "totss"     "withinss"
[5] "tot.withinss" "betweenss"  "size"     "iter"
[9] "ifault"

The results show:
1. the cluster centroids;
2. the cluster to which each plot is assigned;
3. the within-cluster sum of squares for each cluster. This shows how compact is each cluster – some may be better defined than others;
4. the proportion of the total sum-of-squares (variability among plots) not explained by the clusters, i.e., by the between-cluster sum-of-squares. If any cluster is too diffuse this may indicate that more clusters should have been requested.

Q146 : How much of the total variability is explained by this clustering? What do you conclude from this about (1) the distribution of yields; (2) the choice of number of feature-space clusters? Jump to A146 •

Task 144 : Display a scatterplot of straw vs. grain, with the assigned clusters shown by colour.

We add the cluster assignment to the spatial object’s data frame, and use this to colour the plot.

> mhw.sp$cluster6 <- as.factor(gs.cluster$cluster)
> ggplot(data=mhw.sp@data, aes(grain, straw, color=cluster6)) +
Q147: Do the clusters match your answer from the previous question? Which cluster(s) are most compact? most diffuse? **Jump to A147**

Now we want to see if these feature space zones have a geographic expression; if so, these could be managed separately.

Task 145: Display a map of the field with the plots coloured by feature-space cluster.

We first create the plot and save it in workspace variable `g`, and then display it. This allow us to update the plot with extra or changed information, see below.

```r
> g <- ggplot(mhw.sp@data, aes(c, r, color=cluster6)) +
  +  geom_point(shape=15, size=5)
> print(g)
```
This map is difficult to read because of the “soft” colours. We can find more suitable palettes defined in the **RColorBrewer** package.

**Task 146**: Display the pre-defined palettes for classified (factor) values provided by the **RColorBrewer** package.

We use the `display.brewer.all` function, with the `type` argument to show just the palettes that apply to qualitative values.

```r
> require(RColorBrewer)
> display.brewer.all(type="qual")
```

---

**Task 147**: Repeat the map display, using a stronger palette.

The `scale_colour_brewer` function of **ggplot2** specifies an **RColorBrewer** palette. Notice how the previous plot, saved in workspace variable `g`, is **updated** with this additional specification; there is no need to repeat the code for the original plot.

```r
> g + scale_colour_brewer(name="cluster6", palette="Accent")
```
Now the zones are much clearer.

Q148: Do the feature-space clusters also cluster in geographic space? Would this map be useful for management by zones? Jump to A148

The obvious question is how many management zones to specify. These should be (1) relatively homogeneous in feature space, (2) relatively large and compact in geographic space. In the present case we see that both conditions are not well-met, perhaps specifying fewer zones would give a better result?

Challenge: Repeat the analysis for a different number of proposed zones and compare the results.

20.1 Answers

A145: Of course this is subjective, but I can see (1) very low grain/straw (upper-left of the plot), (2) very high grain/straw (lower right), (3) very high yields of both (upper right), (4) very low yields of both (lower left), (5) and (6) two groups of intermediate yields near the centre of the plot, one lower than the other. Return to Q145

A146: Only 12% of the variation is explained. This means that the clusters are all diffuse and not well-separated. More clusters would of course increase the variation
explained, but might not correspond to useful distinctions.  

**A147**: No very low grain/straw (upper-left of the plot) was identified; the low grain/straw are split, some go with cluster 1 (high yield) and some with cluster 6 (medium-high yield). The proposed two intermediate clusters were split into three. The highest and lowest yield clusters are the most diffuse.  

**A148**: There is definite clustering in geographic space; however the “zones” generally do not extend beyond about six plots, and there are many single-plot “zones”. This map would not be useful for management by zones.  

### 21 Periodicity*

Mercer & Hall asserted that the field was uniform; however others [35, 43, 49] have suggested that the field was not as uniform as assumed, and that there were systematic patterns of previous management that affected yield. In §9 we determined that the field halves had significantly different straw yields; in §17.4 we determined that the two halves had different spatial structures, i.e., local spatial dependence. These two findings support the idea that the two halves of the field had in the past been managed differently. However, there is another possible management effect: periodic variation across rows or columns due to previous ploughing or ridging. Had this field previously been used with a ridge-furrow system or divided into ‘lands’?  

**Note**: There is also some evidence of periodicity in the analysis of local spatial structure of grain yields in §17.2. The variogram shows a dip at the 5th bin, at about 11.6 m, then an increase in the 6th (13.9 m), then again a dip in the 7th (16.4 m).  

Here we follow the analysis of McBratney and Webster [35] to investigate this possibility. The hypothesis is that in either the W–E (column-wise) or N–S (row-wise) direction that there is either positive or negative autocorrelation in grain or straw yields, at some spacing between columns or rows.  

**Note**: The field layout is explained in §A.  

To investigate this, we use the tools of one- and two-dimensional spatial correlation analysis. Thus we analyze the W–E dimension (columns along single rows) and the N–S dimension (rows along single columns), in all combinations. The 1D and 2D spatial correlations are symmetric; the same correlation is found E–W and W–E, and the same S–N as N–S; for the 2D case the correlations are radially symmetric.  

#### 21.1 Visualizing periodicity

In this section we attempt to reproduce Fig. 3 from McBratney and Webster [35], which shows an autocorrelation surface of grain yields, i.e., autocorrelations at all combinations of row and column lags. Autocorrelation is closely related to semivariance (see Eq. 21.1, just below), and fortunately
a semivariogram surface can be computed by the `variogram` “compute empirical variogram” method of the `gstat` package, specifying the `map` argument as `TRUE`; we apply this to a spatial version of the dataset, with coordinates specified by row and column; i.e., we compute lags in terms of rows and columns:

```r
> mhw.rc <- mhw
> coordinates(mhw.rc) <- ~r + c
> v.map <- variogram(grain ~ 1, loc = mhw.rc, map = TRUE, +   cutoff = 10, width = 1)
> summary(v.map$map$var1)

Min. 1st Qu. Median Mean 3rd Qu. Max. NA's
0.0986 0.1830 0.2040 0.2010 0.2170 0.2620 1

> class(v.map)
[1] "variogramMap" "list"
> plot(v.map, col.regions = bpy.colors(64), xlab = "Row-wise (N-S) lag", +   ylab = "Column-wise (W-E) lag")
```

To convert semivariances $\gamma$ to covariances $C$, and then to correlations $\rho$, we use the relations:

$$C(h) = C(0) - \gamma(h)$$ (21.1)
$$\rho(h) = \frac{C(h)}{C(0)}$$ (21.2)

By definition the autocorrelation at the origin is 1. The covariance at a point $C(0)$ is estimated as as the variance over the field, using the `var` function. We also replace the “not applicable” semivariance at the origin with the known autocorrelation, i.e., 1, finding the proper location in the matrix with the `is.na` and `which` functions.
c0 <- var(mhw$grain)
v.map$map$cov <- c0 - v.map$map$var1
v.map$map$cor <- v.map$map$cov/c0
v.map$map$cor[which(is.na(v.map$map$cor))] <- 1
summary(v.map$map$cor)

To view as a 3D plot, we convert the autocorrelations to a matrix using the matrix function, since this is the form required by the wireframe function. To get a perspective view, we use aspect (vertical and horizontal axis ratio) and screen (rotation and tilt) arguments.

str(v.map$map@data)

To view as a 3D plot, we convert the autocorrelations to a matrix using the matrix function, since this is the form required by the wireframe function. To get a perspective view, we use aspect (vertical and horizontal axis ratio) and screen (rotation and tilt) arguments.

> str(v.map$map@data)
'data.frame': 441 obs. of 4 variables:
$ var1 : num 0.202 0.189 0.205 0.19 0.179 ...
$ np.var1: num 150 165 180 195 210 225 240 255 270 285 ...
$ cov : num 0.00776 0.02126 0.00469 0.01954 0.03102 ...
$ cor : num 0.037 0.1012 0.0223 0.0931 0.1477 ...

> n <- sqrt(length(v.map$map$cor))
v.map.mat <- matrix(v.map$map$cor, nrow=n, ncol=n)
> plot(wireframe(v.map.mat, drape=T, aspect=c(1,.25),
+ screen=c(z=225, x=-60), ylab="Column-wise (W-E) lag",
+ xlab="Row-wise (N-S) lag", zlab="rho(h)",
+ main="Autocorrelation surface, grain yields",
+ col.regions=bpy.colors(72)))
This plot is diagonally symmetric. The front (positive, row-wise) half resembles Fig. 3 from McBratney and Webster [35], except at the (+10, +10) corner where our figure shows a strong low autocorrelation (corresponding to a large semivariance in the variogram map) and the corresponding figure has another positive peak.

Q149: Does there appear to be any periodicity in one or both directions? What does this imply about the uniformity of the field? Jump to A149

Task 148: Plot the average autocorrelation along rows and columns.

The variogram map (surface) has this information in the central row and column of the matrix. Since the map is symmetric, we only need to plot from the centre out.

First, compute the correlations along the central row (E–W) and column (N–S):

```r
> (c <- ceiling(n/2))
[1] 11
> (v.map.mat[c:n, c])
[1] 1.00000 0.53053 0.41881 0.37809 0.36462 0.30621 0.20680 0.17216
[9] 0.20485 0.18515 0.15488
> (v.map.mat[c, c:n])
[1] 1.00000000 0.29431834 0.14698574 0.17645418 0.08599151
[6] 0.00029433 0.07806734 0.04102077 -0.13276224 -0.00651483
[11] -0.02592755
```

Plot the autocorrelations in the two directions:

```r
> par(mfrow = c(1, 2))
> str(v.map.mat)
num [1:21, 1:21] 0.037 0.1012 0.0223 0.0931 0.1477 ...  
> plot(v.map.mat[c, c:n], type = "h", ylim = c(-0.2, 1),
+ main = "Along columns (E–W)" , ylab = expression(rho),
+ col = "blue", xlab = "lag (rows)"
)  
> abline(h = 0)
> plot(v.map.mat[c:n, c], type = "h", ylim = c(-0.2, 1),
+ main = "Along rows (N–S)" , ylab = expression(rho),
+ col = "darkgreen", xlab = "lag (columns)"
)
> abline(h = 0)
> par(mfrow = c(1, 1))
```
21.2 Spectral analysis

McBratney and Webster [35] also examine the spatial autocorrelation in the frequency domain (as opposed to the spatial domain); that is, they consider the signal as a sum of periodic components (so-called Fourier analysis), and examine the power spectrum, that is, the relative contribution of each period to the overall signal. This should reveal the periodicity. This technique is widely used in time-series analysis, but is equally applicable to data organized in space. The frequency is relative to a cycle, which in time series is some natural cycle such as a year or day. Here the cycle is the single field.

21.2.1 Theory

The theory of power spectra and the Fourier transform from and to the spatial domain are explained in the text of Webster and Oliver [53, Ch. 7]; we present enough theory here to motivate and interpret our application. We consider the 1D case, i.e., observations arranged in a line, to develop the theory, and then extend it to the 2D case.

The fundamental transformation from covariances at each lag $C(h)$ to the power at each frequency $R(f)$ is:

$$ R(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(fh) C(h) dh $$  \hspace{1cm} (21.3)  

In words, the integral is of the covariances, each multiplied by the cosine at the given frequency, over all lags. As the frequency increases, the period of the cosine function gets shorter. If the higher covariances coincide with that period, the power at that frequency is higher.

Eqn. 21.3 assumes a 2nd-order stationary process in $\mathbb{R}^1$, in which as the lag $h$ gets wider, the covariance $C(h)$ approaches 0.

This ideal equation can not be computed, for several reasons. First, we do not have an infinite number of lags (i.e., infinite-length sequence of ob-
servations), so the integral is restricted to some window $L$, after which the covariances $\approx 0$. Second, the experimental covariances are increasingly unreliable at large lags, so a rule of thumb is to limit $L$ to about one-fifth of a 1D series length. Finally, observations are not made at infinitesimal lags, rather they have a minimum spacing, and higher-frequency components, if any, can not be estimated.

So, the spectrum is estimated for frequencies $f$, ranging from $-0.5$ to $0.5$ cycles, from the empirical covariances $\hat{C}(k)$ estimated from the observations. So the spectrum is estimated as:

$$\hat{R}(f) = \frac{1}{2\pi} \left\{ \hat{C}(0) + 2 \sum_{k=1}^{L-1} \hat{C}(k) \cos(\pi fk) \right\}$$

(21.4)

Notice that the integral of Eqn. 21.3 is replaced by a sum over the chosen window. For correlations rather than covariances, replace $C$ with $c$ and omit the term with $\hat{C}(0)$. These sums estimate the relative contribution, also called spectral density or power, of each frequency to the overall signal in the 2nd-order stationary series.

The cosine term varies from 0 to $k$ for $f = 0...0.5$, i.e., $\pi f = 0...\pi/2$. For example, at $f = 0$ (i.e., at the centre of a full cycle), in terms of correlations $c$:

$$\hat{R}(0) = \frac{1}{\pi} \sum_{k=1}^{L-1} \hat{c}(k)$$

At $f = 0.5$ (i.e., half a cycle):

$$\hat{R}(0.5) = \frac{1}{\pi} \sum_{k=1}^{L-1} \hat{c}(k) \cos\left(\frac{\pi}{2}k\right)$$

There is a further refinement to the estimate of Eq. 21.4: using a window that emphasizes the more reliable shorter lags, without discarding too much information from longer lags. Webster and Oliver [53, §7.3.1] propose several window functions $w(k)$ to multiply the covariances in Equation 21.4; the one used by McBratney and Webster [35, Eq. 9] is the Bartlett window:

$$w(k) = \begin{cases} 1 - (|k|/L) & \text{if } 0 \leq |k| \leq L \\ 0 & \text{if } |k| > L \end{cases}$$

(21.5)

Note: Fig. 7.3 of Webster and Oliver [53] shows the relative weights of lags within a window for the rectangular, Bartlett, and Parzen window; these authors favour the latter.

There is only one quantity left to estimate; this is the auto-covariance at each lag $h$, including zero; this follows directly from the definition of covariance:

$$\hat{C}(h) = \frac{1}{N-h} \sum_{i=1}^{N-h} \{ (z(i) - \bar{z})(z(i+h) - \bar{z}) \}$$

(21.6)

where $z$ is the data value.

We now apply this theory to the 2D case of the Mercer & Hall wheat field.
We first compute the auto-covariances \( c_{p,q} \) in two dimensions as [35, Eq. 6]:

\[
\begin{align*}
   c_{p,q} &= \frac{1}{(m-p)(n-q)} \sum_{i=1}^{(m-p)} \sum_{j=1}^{(n-q)} (x_{i,j} - \bar{x})(x_{i+p,j+q} - \bar{x})
\end{align*}
\] (21.7)

where \( \bar{x} \) is the mean of the \( m \times n \) matrix, and the lags in the two dimensions are \( p \) and \( q \). Thus, all possible combinations cells that contribute to the two-dimensional lag are included in the weighted sum. Auto-correlations are obtained by dividing the auto-covariances by \( c_{0,0} \), i.e., the overall variance.

Note: This should give identical results to the variogram surface, converted to an auto-correlation surface, which was computed by the variogram method in §21.1, above. However, its calculation is a good example of the use of the for flow control operator.\(^{23}\)

To compute this sum, we use the for flow control operator in a so-called “for-loops”. These are not much used in R, since in many cases vectorized operators can be used, but in this case we need to step through the matrix, and at each position compute a covariance of selected neighbour cells.

We first convert the grain yields to a matrix matching the field shape, and centre the yields on the mean, since the covariances are based on differences, not absolute values:

```r
> m <- max(mhw$r)
> n <- max(mhw$c)
> str(mhw.grain.matrix <- matrix(data = (mhw.rc$grain - mean(mhw.rc$grain)), nrow = m, ncol = n))
```

num [1:20, 1:25] -0.3186 0.1214 0.5614 -0.0486 -0.3186 ...

---

**Task 149**: Write a function to implement Eqn. 21.7.

The following functions compute covariances \( c_{p,q} \) and \( c_{p,-q} \), respectively, when given the the row and column lags, and the data matrix, as arguments. The return function returns the computed covariance to the caller.

```r
> cpq <- function(p, q, mat) {
+   s <- 0
+   for (i in 1:(m - p)) for (j in 1:(n - q)) s <- s +
+     (mat[i, j]) * (mat[i + p, j + q])
+   return((1/((m - p) * (n - q))) * s)
+ }

> cpmq <- function(p, q, mat) {
+   s <- 0
+   for (i in 1:(m - p)) for (j in (q + 1):n) s <- s +
+     (mat[i, j]) * (mat[i + p, j - q])
+   return((1/((m - p) * (n - q))) * s)
+ }
```

\(^{23}\) This was covered in more detail in §15.
Task 150: Apply these functions the desired combination of row and column lags.

Following McBratney and Webster [35] we compute up to fourteen lags in both directions, although this is considerably more than the recommended one-fifth of the overall dimension (i.e., 4 rows and 5 columns). The matrix is radial symmetric, so we can fill in the other two quadrants from the first two.

```r
> max.l <- 14
> d <- 2 * max.l + 1
> ch <- matrix(0, d, d)
> for (lag.r in 1:max.l) for (lag.c in 1:max.l) {
+   ch[(max.l + 1) + lag.r, (max.l + 1) - lag.c] <- cpmq(lag.r, lag.c, mhw.grain.matrix)
+   ch[(max.l + 1) - lag.r, (max.l + 1) + lag.c] <- ch[(max.l + 1) + lag.r, (max.l + 1) - lag.c]
+ }
> for (lag.r in 0:max.l) for (lag.c in 0:max.l) {
+   ch[(max.l + 1) + lag.r, (max.l + 1) + lag.c] <- cpq(lag.r, lag.c, mhw.grain.matrix)
+   ch[(max.l + 1) - lag.r, (max.l + 1) - lag.c] <- ch[(max.l + 1) + lag.r, (max.l + 1) + lag.c]
+ }
> ch <- ch/var(mhw$grain)
> summary(as.vector(ch))
```

```
Min. 1st Qu.  Median   Mean 3rd Qu. Max. 
-0.45000 -0.10400 -0.01260 -0.00802 0.09190 0.99800
```

Task 151: Plot the autocorrelation surface.

Again we use the `wireframe` function:

```r
> plot(wireframe(ch, drape=T, aspect=c(1,.25),
+    screen=c(z=225, x=-60), ylab="Column-wise (W-E) lag",
+    xlab="Row-wise (N-S) lag", zlab="rho(h)",
+    main="Autocorrelation surface, grain yields",
+    auto.key=T,
+    col.regions=bpy.colors(72)))
```
Indeed, this is the same surface computed in §21.1, but extended to maximum lag 14, so the stretch is somewhat different, here reaching $-0.45$ at lag $(-14, +14)$. The periodicity in the N-S direction is obvious.

### 21.2.3 Power spectrum

The 1D spectral calculation of Eqn. 21.4 can be extended into 2D, by considering the lags, correlations, and weights in 2D. For frequencies from $-0.5$ to $0.5$ cycles per sampling interval, the 2D power spectrum (in the frequency domain) is estimated from the auto-covariances (in the spatial domain) as [35, Eq. 8]:

$$G_{r,s} = K^{-1} \sum_{q=-L}^{L} \sum_{p=-L}^{L} c_{p,q} w_{p,q} \cos \left\{ \left( \frac{\pi}{T} \right) (rp + sq) \right\}$$  \hspace{1cm} (21.8)

where:

- $K$ is a normalizing constant, taken here as $(2\pi)^2 = 4\pi^2$;
- $L$ is the maximum lag, i.e., window size, to be included in the estimate;
- $p, q$ are the lags in two dimensions (rows or columns);

\[24\] This constant only affects the absolute values, not the relative magnitudes, which are of primary interest.
• $r, s = -T, -T + 1, \ldots, 0, 1, T - 1, T$, i.e., the number of frequency estimates in the interval $-0.5 \ldots 0.5$ cycles;

• $w_{p,q}$ is Bartlett’s smoothing function (Equation 21.5), which damps the cosine function’s amplitude at greater lags, computed in 2D as:

$$w_{p,q} = \begin{cases} 
1 - |h|/L & \text{if } 0 \leq |h| \leq L \\
0 & \text{if } |h| > L 
\end{cases}$$

where $|h| = \sqrt{p^2 + q^2}$, i.e., the Euclidean distance between plots, and $L$ is the maximum lag at which we want compute the spectral density.

That is, the spectral estimate for a given combination of frequencies is a weighted sum of lagged covariances multiplied by the appropriate point in the period, i.e., the argument to the cosine function.

The number of frequency estimates $T$ is not dependent on the number of lags; it is set by the analyst to obtain a sufficiently fine-resolution estimate. There is no disadvantage to a high value of $T$ other than computation time; McBratney and Webster [35] used $T = 50$ to obtain their Fig. 5.

However, the choice of $L$ is critical: small $L$ are reliable but do not reveal high-frequency components; large $L$ may be unreliable. The usual procedure is to compute for a variety of $L$ and examine the spectra to see where sufficient detail, without excessive noise, is found. It is also possible to compute confidence intervals; see Webster and Oliver [53, §7.3.3]; we have not (yet) implemented that here.

Task 152: Write a function to implement Eqn. 21.8, i.e., to convert the autocorrelation surface into a spectrum $G_{r,s}$ for each combination of frequency estimates, from $-t, -t + 1, \ldots, 0, 1, \ldots t - 1, t$, where $t$ is the number of frequency estimates.

We use the function command to write a function from the lag combination $r$ and $s$, the number of frequency estimates $t$, the maximum lag $L$, and the matrix of correlation coefficients cor.mat:

Note: This function also includes an internal function (i.e., only visible inside the function) to compute Bartlett’s weighting.

```r
> grs <- function(r = 0, s = 0, t, L, cor.mat) {
+   w <- function(x, y) {
+     h <- sqrt(x^2 + y^2)
+     return(ifelse(h <= L, 1 - (h/L), 0))
+   }
+   max.p <- dim(cor.mat)[1]
+   max.q <- dim(cor.mat)[2]
+   centre.p <- ((max.p - 1)/2) + 1
+   centre.q <- ((max.q - 1)/2) + 1
+   sum <- 0
+   for (q in -L:L) {
+     if (centre.q + q + 1 > max.q)
```
for (p in -L:L) {
    if (centre.p + p + 1 > max.p)
        break
    s1 <- (cor.mat[centre.p + p, centre.q + q] *
           w(p, q) * cos((pi/t) * ((r * p) + (s * q))))
    sum <- sum + s1
}

return(sum/(4 * pi^2))

Task 153: Apply this function at a resolution of 50 estimates per cycle, first along the 1D W-E axis, with no N-S offsets and for a window size $L = 10$.

Since the spectrum is symmetric, we only need to compute the positive half, i.e., from $s = 0...T$:

```r
theta <- 50
dens <- rep(0, theta + 1)
for (s in 0:theta) dens[s + 1] <- grs(0, s, t = theta,
+ L = 10, cor.mat = ch)
```

Task 154: Plot the spectral density as a function of frequency.

We use the interpolating spline function `spline` to smooth the graph. However, the estimates at each knot are not changed by the spline:

```r
plot(dens ~ seq(0, 0.5, length = theta + 1), type = "p",
+ main = "Spectral density, W-E", sub = "window size 10",
+ ylab = "density", xlab = "frequency, cycles", pch = 20,
+ cex = 0.6)
dens.smooth <- spline(dens)
lines(dens.smooth$y ~ seq(0, 0.5, length = length(dens.smooth$x)),
+ lty = 1)
grid()
```
The selection of the window width $L$ is subjective, so we now look for the width that best reveals the spectral characteristics.

**Task 155**: Compute the power spectrum along the E-W dimension for window sizes $L = 4, 6, 8, 10, 12, 14$ and plot their spectral densities vs. frequency on one graph.

We set up a matrix whose rows are the window size and whose columns are the densities for that window size, at the chosen resolution:

```r
dl.s <- seq(4,14,by=2)
theta <- 50
dens <- matrix(rep(0, length(l.s)*(theta+1)),
+    nrow=length(l.s))
for (i in 1:length(l.s)) {
    for (s in 0:theta) {
        dens[i, s + 1] <- grs(0, s, theta, l.s[i], ch)
    }
}
```

We then compute the spectral density for each window size, again using a `for` loop to step through the window sizes, i.e., rows of the results matrix:

```r
> for (i in 1:length(l.s)) {
+    dens.smooth <- spline(dens[i, ])
+    lines(dens.smooth$y ~ seq(0, 0.5, length = length(dens.smooth$x)),
+        lty = i)
```
Note: This figure is slightly different from Fig. 5 in McBratney and Webster [35]; the reason is not clear. However, the major features are similar.

Q151: What are the outstanding features of the power spectrum? What do they imply about periodicity in this direction? Jump to A151

Q152: Which window size best reveals the features? Jump to A152

In §21.1 we concluded that there was periodicity in the W–E direction (column-wise along rows) but not the N–S direction (row-wise along column); the W–E power spectrum confirms the first; now we examine the second.

Task 156: Compute and plot the N–S power spectrum for window width $L = 10$.

```r
> theta <- 50
> dens <- rep(0, theta + 1)
> for (r in 0:theta) dens[r + 1] <- grs(r, 0, t = theta, L = 10, cor.mat = ch)
> plot(dens ~ seq(0, 0.5, length = theta + 1), type = "p",
+ main = "Spectral density, N–S", sub = "window size 10",
+ ylab = "density", xlab = "frequency, cycles", pch = 20,
+ cex = 0.6)
```
> dens.smooth <- spline(dens)
> lines(dens.smooth$y ~ seq(0, 0.5, length = length(dens.smooth$x)),
+     lty = 1)
> grid()

Spectral density, N–S

Q153: Is there any evidence of periodicity in this direction? Jump to A153

We can visualize the power in both directions simultaneously, and also discover if there are any interactions of the directions; for example if the periodicity were not along rows or columns.

Task 157: Compute and plot the two-dimensional power spectrum for window width $L = 10$; this corresponds to Fig. 7 in McBratney and Webster [35].

This requires $O(\theta^2)$ density calculations, so we reduce the frequency resolution to 25 to speed up the computation:

> theta = 25
> dens <- matrix(rep(0, (theta + 1)^2), nrow = theta +
+     1)
> for (s in 0:theta) for (r in 0:theta) dens[r + 1, s +
+     1] <- grs(r, s, theta, 10, ch)
> wireframe(dens, drape = T, aspect = c(1, 0.35), screen = c(z = 225,
+     x = -60), xlab = "N–S frequency", ylab = "E–W frequency",
+     zlab = "density", auto.key = T, col.regions = topo.colors(72))
Task 158: Remove the variables created in this section.

```r
> rm(mhw.rc, v.map, c0, v.map.mat, m, n, c, max.l, lag.r,
+    lag.c, r, s, theta, dens, dens.smooth, l.s, grs,
+    ch)
```

Challenge: Repeat this analysis for straw yields. Does this confirm, weaken, or strengthen the conclusions from the analysis of grain yields?

Challenge: Recall (§17.4) that the local spatial structure is different in the N and S field halves. Is the periodicity in the E–W direction present in both field halves? If so, does it show the same pattern?

Challenge: Use anisotropic variogram analysis to confirm the periodic effect.

21.3 Answers

**A149:** There are clear “ripples” on the autocorrelation surface in the E–W direction (columns along rows); these appear to be at 3-plot intervals. However, this pattern varies somewhat with increasing separation in the N–S direction. There seems to be no such a pattern in the N–S direction, just the expected decreasing autocorrelation with separation.

*Return to Q149*

**A150:** The autocorrelation graphs show clearly the pattern of the previous answer, i.e., along rows there is a clear dip in autocorrelation at lags 3 and 6, and even
a negative autocorrelation at lag 9. No such pattern is seen along columns. This implies a periodic structure along the E-W (row-wise) direction, at an interval of approximately three rows; recalling that the row length is 3.30 m (§A), the periodicity appears to have a fundamental period of about 10 m; this is also the conclusion of McBratney and Webster [35, §4], who examine the likely causes. Return to Q150

A151: The highest power is at the origin, i.e., the overall mean, representing a full cycle. However there is significant power at approximately 0.33 cycles, i.e., $20/3 = 6.6$ plots. Return to Q151

A152: $L = 10$ is sufficiently detailed to clearly see the spike at 0.33 cycles without the small fluctuations for $L > 10$. At $L < 4$ the feature becomes increasingly vague. Return to Q152

A153: No. Return to Q153

22 The effect of plot size*

Mercer and Hall’s original research objective was to determine how within-plot variability is affected by plot size. To investigate this, they grouped the 1 acre field into plots of 1/500 acre (the original plots), 1/250 acre, 1/125 acre, 1/50 acre, 1/25 acre, and finally 1/10 acre; they measured the variability in the resulting samples and graphed this by plot size. They then could determine how large a plot would be necessary to reduce the variability to an acceptable level. We will repeat their analysis here.

Q154: Based on the geostatistical analysis (§17.2), what size plot would be expected to remove most of the local variation? Jump to A154

Before proceeding, we need an operational definition of heterogeneity, so we can compare the variability between different plot sizes. In §7 we used the probable error, but that required modelling a normal distribution. A simpler, and commonly-used, measure of variability for samples that are approximately normally-distributed is the coefficient of variation (CV), which is defined as:

$$CV = \frac{s}{\bar{x}}$$

This normalizes the sample standard deviation $s$ by the sample mean $\bar{x}$. It is commonly expressed in percent. This measure was also used by Mercer and Hall.

There is no R function to compute the CV; we can compute it on any sample by first using the mean function and then the sd function, and then taking their ratio:

25 although they mistakenly refer to it as “standard deviation”
To avoid writing this out for each vector whose CV we want to compute, we can write a small function to do this computation on any sample; this uses the function function to define the algorithm.

**Task 159**: Write a function to compute the CV of a vector.

```r
> cv <- function(x) {
+   round(100 * sd(x)/mean(x), 1)
+ }
> class(cv)
[1] "function"
```

The object cv in the workspace is a function which can now be applied to any vector, just like a built-in R function.

**Q155**: What is the CV of the grain yields of the entire set of 500 plots?  

```r
> cv(mhw$grain)
[1] 11.6
```

Now we group the plots into increasingly-larger plots and see how the CV of the set is affected. Mercer and Hall compared six sizes: 1/250, 1/125, 1/100, 1/50, 1/25 and 1/10 acre; these are all possible groupings of rows and columns to this size, given the field layout.

**Task 160**: Determine how adjacent plots can be grouped in increasingly-large blocks, up to 1/10 acre.

We are restricted to divisors of 20 (rows), i.e. 2, 2 and 5, and 25 (columns), i.e. 5:

- 1/500 acre: Original layout; grid is (20 \times 25);
- 1/250 acre: Combine the plots in each two adjacent rows; resulting grid is (10 \times 25);
- 1/125 acre: Combine the plots in each four adjacent rows; resulting grid is (5 \times 25);
- 1/100 acre: Combine the plots in each five adjacent columns; resulting grid is (20 \times 5);
- 1/50 acre: Combine the plots in each five adjacent columns and two adjacent rows; resulting grid is (10 \times 5);
1/25 acre: Combine the plots in each five adjacent columns and four adjacent rows; resulting grid is (5 x 5);

1/10 acre: Combine plots in each five adjacent columns and ten adjacent rows; resulting grid is (2 x 5);

Three larger sizes are possible: 1/5, 1/4 and 1/2 acre; however these do not have enough plots to evaluate variability.

Task 161: Make a data structure with each plot size and its dimensions.

From §16.2 we have the plot length, width, and area:

```r
> plot.len; plot.wid; plot.area
[1] 3.1807
[1] 2.5446
[1] 8.0937
```

We can use these to compute dimensions for each combinations.

We first make a data frame with information about the combinations, using the `data.frame` function to combine three lists, each made with the `c` function; we also name the rows with the `row.names` function for easy reference.

We name each field (column) explicitly with the `field.name = ...` syntax.

```r
> plots <- data.frame(acre.fraction = c(500, 250, 125, + 100, 50, 25, 10), adj.rows = c(1, 2, 4, 1, 2, 4, + 10), adj.col = c(1, 1, 1, 5, 5, 5, 5))
> row.names(plots) <- c("1/500", "1/250", "1/125", "1/100", + "1/50", "1/25", "1/10")
> str(plots)
'data.frame': 7 obs. of 3 variables:
$ acre.fraction: num 500 250 125 100 50 25 10
$ adj.rows : num 1 2 4 1 2 4 10
$ adj.col : num 1 1 5 5 5 5
```

Now we can compute dimensions and add them to the frame with the `cbind` function:

```r
> plots <- cbind(plots, len = plot.len * plots$adj.row)
> plots <- cbind(plots, wid = plot.wid * plots$adj.col)
> plots <- cbind(plots, area = plots$len * plots$wid)
> plots
   acre.fraction adj.rows adj.col len wid area
1/500 500 1 1 3.1807 2.5446 8.0937
1/250 250 2 1 6.3615 2.5446 16.1874
1/125 125 4 1 12.7230 2.5446 32.3748
1/100 100 1 5 3.1807 12.7230 40.4686
1/50 50 2 5 6.3615 12.7230 80.9371
1/25 25 4 5 12.7230 12.7230 161.8742
1/10 10 10 5 31.8075 12.7230 404.6856
```
The row names are shown when the entire frame is printed; this allows us to identify each combination.

**Q156**: What are the dimensions of these in meters, and their areas in m\(^2\)?

**Task 162**: Compute the grain yields for the 1/250 acre plots made up of two adjacent rows, and its CV.

To group in pairs, we make use of the **modulus** arithmetic operator \( \%\% \) to identify the odd and even rows:

```r
> head(mhw$r%%2, 20)
[1] 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0
```

Now we split the plots into two groups using the `unstack` function and sum the two adjacent plots:

```r
> tmp <- unstack(mhw, grain ~ r%%2)
> str(tmp)
'data.frame': 250 obs. of 2 variables:
 $ X0: num 4.07 3.9 3.16 3.42 3.4 4.43 4.46 5.13 4.38 3.61 ...
 $ X1: num 3.63 4.51 3.63 3.18 3.97 3.39 4.52 3.46 4.23 3.85 ...
> grain.250 <- tmp$X0 + tmp$X1
> rm(tmp)
> str(grain.250)
num [1:250] 7.7 8.41 6.79 6.6 7.37 7.82 8.98 8.59 8.61 7.46 ...
> cv(grain.250)
[1] 10.1
```

**Q157**: Is the variation reduced, as expected, when plot size is doubled? By how much?

We now build a data frame of the combined plots, using the `data.frame` function, labelling each with its original column number and the average of the two rows:

```r
> plots.250 <- data.frame(r = seq(1.5, 19.5, by = 2), c = rep(1:25, 10), grain = grain.250)
> str(plots.250)
'data.frame': 250 obs. of 3 variables:
 $ r : num 1.5 3.5 5.5 7.5 9.5 11.5 13.5 15.5 17.5 19.5 ...
 $ c : int 1 1 1 1 1 1 1 1 1 1 ...
 $ grain: num 7.7 8.41 6.79 6.6 7.37 7.82 8.98 8.59 8.61 7.46 ...
```
Task 163: Visualise the variation across the field with the 1/250 acre plot size.

We visualize by colour ramp:

```r
> with(mhw,
+ plot(plots.250$c, plots.250$r, pch=20, cex=2,
+     bg="lightblue", xlab="column", ylab="row",
+     main="Grain yield of 1/250 acre plots",
+     sub="Colour of circles from low yield (green) to high (gray)",
+     xlim=c(1, 25), ylim=c(20, 1),
+     col=terrain.colors(8)[cut(grain,
+         quantile(grain, seq(0, 1, length=9)),
+         include.lowest=T, labels=F)])
```

These figures can be compared to those for the 1/500 acre plots in §16.1.

Q158: Does the field appear more homogeneous with 250 vs. 500 plots? What about the pattern of spatial dependence?  

Task 164: Repeat this process for the other combinations.

First, for 1/125 acre. We introduce the very useful `apply` function, which applies any other function (here, the `sum`) across an array margin; here the 1 as second argument specifies that the sum is across rows of the matrix. Since the results of `unstack` are organized into a set of rows, this will add the four plots.

```r
> tmp <- unstack(mhw, grain ~ r%%4)
> str(tmp)
```

220
data.frame: 125 obs. of 4 variables:
$ X0: num 3.9 3.42 4.43 5.13 3.61 4.64 3.35 3.7 3.89 4.22 ... 
$ X1: num 3.63 3.63 3.97 4.52 4.23 4.15 4.27 3.61 3.79 3.87 ... 
$ X2: num 4.07 3.16 3.4 4.46 4.38 4.21 3.55 3.71 4.09 4.12 ... 
$ X3: num 4.51 3.18 3.39 3.46 3.85 4.29 3.5 3.64 4.42 4.28 ... 

> grain.125 <- apply(tmp, 1, sum)
> rm(tmp)
> str(grain.125)
num [1:125] 16.1 13.4 15.2 17.6 16.1 ... 

> cv(grain.125)
[1] 8.9

> plots.125 <- data.frame(r = seq(2, 18, by = 4), c = rep(1:25, each = 10), grain = grain.125)
> str(plots.125)
data.frame: 250 obs. of 3 variables:
$ r : num 2 6 10 14 18 2 6 10 14 18 ...
$ c : int 1 1 1 1 1 1 1 1 1 1 ...
$ grain: num 16.1 13.4 15.2 17.6 16.1 ... 

For the 1/100 acre plots the combination is by column:

> tmp <- unstack(mhw, grain ~ c%%5)
> grain.100 <- apply(tmp, 1, sum)
> rm(tmp)
> cv(grain.100)
[1] 7

> plots.100 <- data.frame(r = rep(1:20, each = 5), c = seq(3, 23, by = 5), grain = grain.100)
> str(plots.100)
data.frame: 100 obs. of 3 variables:
$ r : int 1 1 1 1 1 2 2 2 2 2 ...
$ c : num 3 8 13 18 23 3 8 13 18 23 ...
$ grain: num 20 21.1 21.7 20.1 21.2 ... 

The 1/50 acre plots are the first where both rows and columns are combined. So we have to repeat the unstacking process twice. However, we can start from the 1/100 acre frame which already has combined the columns.

> tmp <- unstack(plots.100, grain ~ r%%2)
> grain.50 <- apply(tmp, 1, sum)
> rm(tmp)
> cv(grain.50)
[1] 5.9

> plots.50 <- data.frame(r = rep(seq(1.5, 19.5, by = 2), each = 5), c = seq(3, 23, by = 5), grain = grain.50)
> str(plots.50)

221
The 1/25 acre plots are constructed similarly, but combining four instead of two rows:

```r
> tmp <- unstack(plots.100, grain ~ r%%4)
> grain.25 <- apply(tmp, 1, sum)
> rm(tmp)
> cv(grain.25)
[1] 4.8
```

```r
> plots.25 <- data.frame(r = rep(seq(2.5, 18.5, by = 4),
+ each = 5), c = seq(3, 23, by = 5), grain = grain.25)
> str(plots.25)
'data.frame': 25 obs. of 3 variables:
$ r : num 2.5 2.5 2.5 2.5 2.5 6.5 6.5 6.5 6.5 6.5 ...
$ c : num 3 8 13 18 23 3 8 13 18 23 ...
$ grain: num 79.9 79.8 83.8 84.6 82.4 ...
```

The 1/10 acre plots are constructed similarly, but combining ten rows:

```r
> tmp <- unstack(plots.100, grain ~ r%%10)
> grain.10 <- apply(tmp, 1, sum)
> rm(tmp)
> cv(grain.10)
[1] 3.9
```

```r
> plots.10 <- data.frame(r = rep(seq(5.5, 15.5, by = 10),
+ each = 5), c = seq(3, 23, by = 5), grain = grain.10)
> str(plots.10)
'data.frame': 10 obs. of 3 variables:
$ r : num 5.5 5.5 5.5 5.5 5.5 15.5 15.5 15.5 15.5 15.5 ...
$ c : num 3 8 13 18 23 3 8 13 18 23 ...
$ grain: num 203 204 205 207 202 ...
```

Now we attempt to answer Mercer & Hall’s research question.

Q159: What is the trend of the summary statistics (extremes, mean, median, IQR) as the plot size increases, normalized to a 1/500 acre basis? 

Note: To compare these we have to divide the summary by the number of 1/500 acre plots making up the larger plots:

```r
> summary(mhw$grain)

Min. 1st Qu.  Median  Mean 3rd Qu.  Max. 
2.73  3.64  3.94  3.95  4.27  5.16
```

222
What is the trend in the CV as the plot size increases?

```r
> print(size.cv <- data.frame(area = plots$area, cv = c(
+   cv(mhw$grain), cv(plots.250$grain), cv(plots.125$grain),
+   cv(plots.100$grain), cv(plots.50$grain), cv(plots.25$grain),
+   cv(plots.10$grain))))
```

```
area  cv
1 8.0937 11.6
2 16.1874 10.1
3 32.3748 8.9
4 40.4686 7.0
5 80.9371 5.9
6 161.8742 4.8
7 404.6856 3.9
```

```r
> plot(size.cv$area, size.cv$cv, xlab="Plot size, m^2",
+      ylab="Coefficient of variation, \%",
+      main="Plot size vs. CV, Mercer-Hall grain", type="b",
+      xlim=c(0,600))
> grid()
> text(size.cv$area, size.cv$cv, pos=4,
+      paste(
+        plots$adj.rows,
+        " row",
+      
```
Q161: What plot size do you recommend? Jump to A161 •

We now clean up from this section:

```r
> rm(cv, plot.len, plot.wid, plot.area, plots, plots.250, 
    + plots.125, plots.100, plots.50, plots.25, plots.10, 
    + grain.250, grain.125, grain.100, grain.50, grain.25, 
    + grain.10, size.cv)
```

22.1 Answers

A154: The range of local spatial dependence was about 8 m; plot size is (3.30 m long x 2.45 m wide) (§4A); grouping six plots into one plot of (6.60 m long x 7.35 m wide) would remove most of this structure; grouping 12 plots into one plot (9.90 m long x 9.70 m wide) would remove all of it. Return to Q154 •

A155: 11.6%. Return to Q155 •

A156: The dimensions are:

```r
> plots[, c("len", "wid", "area")]
```
A157: Yes, it is reduced somewhat, from 11.6% to 10.1%. Return to Q157

A158: There are half the plots so the detailed spatial structure is lost. However there are still clear patches of higher and lower yields. Return to Q158

A159: The means are almost identical (3.95 to 3.95), and the medians close (3.94 to 3.98); however the extremes (and hence the range) are reduced as plot size increases (from 2.73...5.16 in the full set to 3.72...4.14 in the largest plot) and the IQR is somewhat narrower (from 3.64...4.27 in the full set to 3.82...4.08 in the largest plot). Return to Q159

A160: The CV decreases rapidly at first, from 11.6% (500 plots) to 10.1% (250 plots) to 8.9% (125 plots) to 7% (100 plots), and then less dramatically, to 5.9% (50 plots), 4.8% (25 plots), and 3.9% (10 plots). The graph has a hyperbolic shape and shows a clear inflection point around 80 m$^2$ plot size (50 plots per acre). Return to Q160

A161: This depends on the precision required, which depends on the purpose of the experiment. However, the apparent inflection point of the CV vs. plot size curve is at two rows, five columns, i.e., plots of 1/50 acres, or about 80 m$^2$: 6.36 m long by 12.72 m wide. Put another way, one acre could be used for a trial of ten treatments (e.g., crop varieties or fertilizer combinations) with five replications, with a CV due to random error (not treatment effects) of 5.9%. Return to Q161
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A Example Data Set

In the early days of scientific agriculture, Mercer and Hall [36] were trying to determine the optimum plot size for agricultural yield trials:

- Plots that are too *small* will be too variable;
- Plots that are too *large* waste resources (land, labour, seed); if the land area is limited, the number of treatments will be unnecessarily small.

So, they performed a very simple experiment: an apparently homogeneous field was selected, prepared as uniformly as possible and planted to the same variety of wheat. They attempted to treat all parts of the field exactly the same in all respects during subsequent farm operations. When the wheat had matured, the field was divided into 500 equally-size plots. Each plot was harvested separately. Both grain and straw were air-dried, then hand-threshed and weighed to a precision of 0.01 lb (= 4.54 g). The reported values are thus air-dry weight, lb plot$^{-1}$.

The field was a square of 1 acre$^{26}$, which is 0.40469 ha or 4,046.9 m$^2$, which was divided into a 20 rows by 25 columns, giving 500 plots, each of 1/500 acre, which is about 8.09 m$^2$ (3.30 m long x 2.45 m wide). We do not have records of the original orientation of the field, so we assume that the rows ran W to E, with 25 plots in each row, beginning at 1 on the W and running to 25 at the E, so that columns run N to S with 20 plots in each, running from 1 at the N to 20 at the S. Thus the NW corner (1,1) is plot 1, the NE corner (1, 25) is plot 481, the SE corner (25, 20) is plot 500, and the SW corner (1, 20) is plot 20.

Research questions  This experiment was one of a series of so-called uniformity trials which were conducted early in the 20$^{th}$ century [7, 14], mainly to determine optimum plot sizes [55], field layouts and numbers of replications$^{27}$.

This data set has attracted many statisticians since 1911 [14, 28, 35, 37, 43, 49, 54] because of a simple fact: although the yields should be identical, they are not; in fact they vary considerably. How is this to be explained?

Mercer and Hall distinguished two possible causes:

"If we consider the causes of variation in the yield of a crop it seems that broadly speaking they are divisible into two kinds. The first are random, occurring at haphazard all over the field. Such would be attacks by birds, the incidence of weeds or the presence of lumps of manure. The second occur with more regularity, increasing from point to point or having centres from which they spread outwards; we may take as instances of this kind changes of soil, moist patches over springs or the presence of rabbit holes along a hedge."$^9$

$^{26}$ 2.471054 acres = 1 ha = 10 000 m$^2$

$^{27}$ The intimate relation between the development of applied statistics and scientific agriculture is given in fascinating detail by Gower [21]
The first we recognize as multiple small random errors, with no spatial pattern, which should be normally-distributed (Gaussian) errors.

The second, however, may be evidence of local spatial autocorrelation, which can be investigated by geo-statistical analysis.

Others [35, 43, 49] have suggested that the field was not as uniform as assumed, so that there are both random effect as mentioned by Mercer and Hall but also systematic effects from previous management. These effects could include a regional trend, management blocks, or periodic effects due to, e.g., ridge-and-furrow or previous use as an orchard.

The CSV file  The data has been prepared as the comma-separated values ("CSV") file mhw.csv in a plain-text editor. The first line gives the four field names:

"r","c","grain","straw"

These represent:

r:  Row number in the field
c:  Column number in the field
grain:  Grain yield, lbs plot\(^{-1}\)
straw:  Straw yield, lbs plot\(^{-1}\)

The following 500 lines each represent a plot; the four fields are separated by commas. For example, the first line is:

1,1,3.63,6.37

If you can not find the CSV file in digital form, here it is to copy-and-paste into a text file, which you should name mhw.csv. Note this is presented here in three columns to save space; the file should be one long column of four fields.

"r","c","grain","straw"
1,1,3.63,6.37
18,1,4.38,6.72
19,1,3.85,6.59
20,1,3.61,6.2
1,2,4.15,6.85
19,2,4.28,7.03
2,2,4.21,7.29
20,2,4.22,7.65
3,2,4.29,7.71
1,3,4.06,7.19
4,2,4.64,8.23
2,3,4.15,7.41
5,2,4.27,7.73
3,3,4.73,35
6,2,3.55,6.45
4,3,4.05,7.89
7,2,3.5,5.87
5,3,4.92,8.58
8,2,3.35,5.71
6,3,4.08,7.04
9,2,3.61,6.01
7,3,4.23,7.02
10,2,3.71,6.29
8,3,4.07,7.05
11,2,3.64,6.3
9,3,4.67,7.64
12,2,3.71,6.29
8,3,4.07,7.05
13,2,3.64,6.3
9,3,4.67,7.64
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B Colours

Colours may be specified in several ways; the most intuitive is by predefined name; these can be listed with the colours or colors methods.

```r
> colours()
```

```
[1] "white"   "aliceblue" "antiquewhite" "antiquewhite1"
[5] "antiquewhite2" "antiquewhite3" "antiquewhite4" "aquamarine"
```

Note: These colours are shown various ways on the PDF colour chart [http://research.stowers-institute.org/efg/R/Color/Chart/ColorChart.pdf](http://research.stowers-institute.org/efg/R/Color/Chart/ColorChart.pdf).

These colours can be visualised as a bar graph:

```r
> plot(seq(1:length(colors())), rep(2, length(colours())), type="h",
+    lwd=2, col=colors(), ylim=c(0,1), xlab="Colour number",
+    ylab="", yaxt="n",
+    main="Colours available with the colour() function")
```

![Colours available with the colour() function](image-url)

An individual colour number can be identified interactively with the identify function; left-click on the vertical colour bar at its midpoint; right-click anywhere in the graph when done.

```r
> abline(h=0.5, lwd=3)
> (selected <- identify(seq(1:length(colors())),
+    rep(0.5, length(colors()))))
> colors()[selected]; rm(selected)
```

For example, clicking on the light blue bar near colour 430, and then right-clicking to end the interaction, shows the colour number and name:

```
[1] 432
```
```
[1] "lightskyblue2"
```

Colours can also be referred by number; this is their position in the active palette. These names are displayed or extracted with the palette function:

```r
> palette()
```
Numbered colours are often used when the graphical element matches a numbered element in some data structure:

```r
> boxplot(mhw$straw ~ mhw$r, col=mhw$r, xlab="row", + ylab="Straw yield, lbs plot-1")
```

Here the row number is directly used as the colour: row 1 black, row 2 red, row 3 green etc. Note that the colours are recycled if there are more plot elements than colours in the palette.

The `palette` function can also be used to set the palette. For example to make a 20-element grey-scale to match the 20 rows of wheat plots:

```r
> palette(gray(seq(0,.9,len=20))); palette()
```

```
[1] "black" "#0C0C0C" "#181818" "gray14" "gray19" "#3C3C3C"
[7] "#484848" "#555555" "gray38" "#6D6D6D" "#797979" "gray52"
[13] "gray57" "#9D9D9D" "darkgray" "gray71" "#C1C1C1" "#CDCDCD"
[19] "gray85" "#E6E6E6"
```

```r
> boxplot(mhw$straw ~ mhw$r, col=mhw$r, + xlab="row", ylab="Straw yield, lbs plot-1")
> palette("default"); palette();
```

```
[1] "black" "red" "green3" "blue" "cyan" "magenta"
[7] "yellow" "gray"
```
Note how some colours are given by their Red-Green-Blue saturations, as two hexadecimal digits per hue, e.g. \#E6E6E6\ is approximately 90% of white:

```r
> as.numeric("0xe6")/as.numeric("0xff")
[1] 0.90196
```

This example also shows the `gray` function to set up a grey-scale colour ramp; other colour ramp functions are `rgb`, `hsv`, `rainbow`, and `heat.colors`.

### Plotting symbols

There are 25 pre-defined plotting symbols which can be specified with the `pch`, `fg` and `bg` graphics arguments:

In addition, ASCII characters can be used; e.g. `pch=51` prints a ‘1’. 

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